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A Sandwich-type Theorem with Applications to Univalent Functions

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Abstract—In the present paper, we obtain a sandwich-type theorem. As applications of our main result, we discuss the univalence and starlikeness of analytic functions in terms of certain differential subordinations and differential inequalities.

Keywords—Univalent function, Starlike function, Differential subordination, Differential superordination.

I. Introduction

Let $\mathcal{H}$ be the class of functions analytic in $\mathbb{E} = \{z : |z| < 1\}$ and for $a \in \mathbb{C}$ (set of complex numbers) and $n \in \mathbb{N}$ (set of natural numbers), let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$.

Let $\mathcal{A}$ be the class of functions f, analytic in $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

Denote by $S^*(\alpha)$ and $K(\alpha)$, respectively, the classes of starlike functions of order α and convex functions of order α, which are analytically defined as follows:

$$S^*(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ 0 \leq \alpha < 1 \right\},$$

and

$$K(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ 0 \leq \alpha < 1 \right\}.$$

We write $S^* = S^*(0)$, the class of univalent starlike (w.r.t. the origin) functions and $K(0) = K$, the class of univalent convex functions.

A function $f \in \mathcal{A}$ is said to be close-to-convex if there is a real number $\alpha, -\pi/2 < \alpha < \pi/2$, and a convex function g (not necessarily normalized) such that

$$\Re\left(e^{i\alpha}\frac{f'(z)}{g'(z)}\right) > 0,\ z \in \mathbb{E}.$$

It is well-known that every close-to-convex function is univalent.

In 1934/35, Noshiro [4] and Warchawski [7] obtained a simple but very interesting criterion for univalence of analytic functions. They proved that if an analytic function f satisfies the condition $\Re(f'(z)) > 0$ for all z in $\mathbb{E}$, then f is close-to-convex and hence univalent in $\mathbb{E}$.

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Let f be analytic in $\mathbb{E}$, g analytic and univalent in $\mathbb{E}$ and $f(0) = g(0)$. Then, by the symbol $f(z) \prec g(z)$ (f subordinate to g) in $\mathbb{E}$, we shall mean $f(\mathbb{E}) \subset g(\mathbb{E})$.

Let $\psi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be an analytic function. If p is an analytic function in $\mathbb{E}$ with $(p(z), zp'(z)) \in \mathbb{C} \times \mathbb{C}$ for all $z \in \mathbb{E}$ and h is a univalent in $\mathbb{E}$, then the function p is said to satisfy first order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z),\ \psi(p(0), 0) = h(0). \qquad (1)$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1). The best dominant is unique up to a rotation of $\mathbb{E}$.

Let $\pi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be analytic and univalent in a domain $\mathbb{C} \times \mathbb{C}$, p be analytic and univalent in $\mathbb{E}$, with $(p(z), zp'(z)) \in \mathbb{C} \times \mathbb{C}$ for all $z \in \mathbb{E}$. Then p is called a solution of the first order differential superordination if

$$h(z) \prec \pi(p(z), zp'(z)),\ h(0) = \pi(p(0), 0). \qquad (2)$$

An analytic function q is called a subordinant of the differential superordination (2), if $q \prec p$ for all p satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2), is said to be the best subordinant of (2). The best subordinant is unique up to a rotation of $\mathbb{E}$.

A number of criteria for univalence and starlikeness of analytic functions are available in literature on univalent functions expressed in terms of certain differential subordinations or differential inequalities. But obtaining new and different criteria for univalence and starlikeness has always been a matter of interest for researchers.

In the present paper, we obtain a sandwich-type theorem and as applications of it, we find some sufficient conditions for univalence and starlikeness of $f \in \mathcal{A}$ in terms of certain differential subordinations and differential inequalities.

We also use our subordination theorem to offer a correction in the result Theorem 1.2, stated below due to Obradovic, et al. [5]. We also extend a result of Obradovic, et al. [5].

Obradovic, et al. [5], obtained univalence and starlikeness of $f \in \mathcal{A}$ in terms of differential operator $1 + \alpha\dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}$. Indeed, they proved the following results.

Theorem 1.1: Let p be analytic in $\mathbb{E}$, $p(z) \neq 0,\ z \in \mathbb{E}$ with $p(0) = 1$ and $\alpha \geq -1/2$. Then

$$\frac{zp'(z)}{p(z)} + \alpha p(z) \prec \alpha\frac{1+z}{1-z} + \frac{2z}{1-z^2} = h(z) \qquad (3)$$

$$\Rightarrow p(z) \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

and $\frac{1+z}{1-z}$ is the best dominant.

Theorem 1.2: Let $f \in \mathcal{A}$ and $\alpha' \in (-\infty, 0) \cup [2/3, \infty)$. Then we have

$$1 + \alpha' \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec G(z) \ \Rightarrow f \in S^*, \qquad (4)$$

where G is the conformal mapping of the unit disc $\mathbb{E}$ with $G(0) = 1$ and $G(\mathbb{E})$

$$= \mathbb{C} \setminus \left\{ w \in \mathbb{C} : \Re(w) = \frac{(1-\alpha')|\alpha'|}{\alpha'}, \right. \\ \left. |\Im(w)| \geq |\alpha'|\sqrt{3 - 2/\alpha'} = \sqrt{3\alpha'^2 - 2\alpha'} \right\}.$$

Remark 1.3: In this remark we shall discuss the procedure of obtaining the above result in Theorem 1.2 by Obradovic, et al. [5]. We face some problems to derive this result by their method. Some details are given below.

In [5], the result in Theorem 1.2 is derived from Theorem 1.1. It is clear that the image of the unit disc $\mathbb{E}$ under h (given in 3), is given by

$$h(\mathbb{E}) = \mathbb{C} \setminus \left\{ w \in \mathbb{C} : \Re(w) = 0,\ |\Im(w)| \geq \sqrt{1+2\alpha} \right\}.$$

By supposing that $\alpha \in [-1/2, \infty) \setminus \{1\}$, they have rewritten the subordination in (3) as

$$\frac{zp'(z)}{|1-\alpha|p(z)} + \frac{\alpha(p(z)-1)}{|1-\alpha|} \prec \frac{h(z)-\alpha}{|1-\alpha|} = H(z) \qquad (5)$$

$$\Rightarrow p(z) \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

where

$$H(\mathbb{E}) = \mathbb{C} \setminus \left\{ w \in \mathbb{C} : \Re(w) = -\frac{\alpha}{|1-\alpha|}, \right. \\ \left. |\Im(w)| \geq \frac{\sqrt{1+2\alpha}}{|1-\alpha|} \right\}.$$

Now by selecting $\alpha' = \frac{1}{1-\alpha}$ and $p(z) = \frac{zf'(z)}{f(z)}$ in (5), they conclude the above result given in Theorem 1.2. But, we observe that with this substitution the differential operator on the left of subordination (5) is not reducing to the differential operator on the left of subordination (4). Therefore the result in Theorem 1.2 can not be concluded as they have done.

We notice that the result can be obtained after cancellation of some factors from either sides of subordination (5). But in that case, a correction is required in $G(\mathbb{E})$, given in Theorem 1.2. The superordinate function G maps the unit disc $\mathbb{E}$ onto the entire complex plane except two slits parallel to the imaginary axis. But we notice that the slits are not being left at the correct place for negative real values of α. This correction has been made in the present paper and the correct form of the result in Theorem 1.2 has been given in Corollary 4.9.

II. Preliminaries

We shall use the following definitions and lemmas to prove our main results.

Definition 2.1: ([2], p.21, Definition 2.2b) We denote by Q the set of functions p that are analytic and injective on $\overline{\mathbb{E}} \setminus \mathbb{B}(p)$, where

$$\mathbb{B}(p) = \left\{ \zeta \in \partial\mathbb{E} : \lim_{z \to \zeta} p(z) = \infty \right\},$$

and are such that $p'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{E} \setminus \mathbb{B}(p)$.

Definition 2.2: A function $L(z,t), z \in \mathbb{E}$ and $t \geq 0$ is said to be a subordination chain if $L(.,t)$ is analytic and univalent in $\mathbb{E}$ for all $t \geq 0$, $L(z,.)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{E}$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 \leq t_2$.

Lemma 2.3: ([6, page 159]). The function $L(z,t) : \mathbb{E} \times [0,\infty) \to \mathbb{C}$, of the form $L(z,t) = a_1(t)z + \cdots$ with $a_1(t) \neq 0$ for all $t \geq 0$, and $\lim_{t \to \infty} |a_1(t)| = \infty$, is a subordination chain if and only if $\Re\left(\frac{z\partial L/\partial z}{\partial L/\partial t}\right) > 0$ for all $z \in \mathbb{E}$ and $t \geq 0$.

Lemma 2.4: ([1]). Let F be analytic in $\mathbb{E}$ and let G be analytic and univalent in $\overline{\mathbb{E}}$ except for points ζ_0 such that $\lim_{z \to \zeta_0} G(z) = \infty$, with $F(0) = G(0)$. If $F \nprec G$ in $\mathbb{E}$, then there is a point $z_0 \in \mathbb{E}$ and $\zeta_0 \in \partial\mathbb{E}$ (boundary of $\mathbb{E}$) such that $F(|z| < |z_0|) \subset G(\mathbb{E})$, $F(z_0) = G(\zeta_0)$ and $z_0F'(z_0) = m\zeta_0G'(\zeta_0)$ for some $m \geq 1$.

Theorem 2.5: ([3]) Let $q \in \mathcal{H}[a,1]$, let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and set $\phi(q(z), zq'(z)) \equiv h(z)$. If $L(z;t) = \phi(q(z), tzq'(z))$ is a subordination chain, and $p \in \mathcal{H}[a,1] \cap Q$, then

$$h(z) \prec \phi(p(z), zp'(z)) \ \Rightarrow q(z) \prec p(z).$$

Furthermore, if $\phi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in Q$, then q is the best subordinant.

III. Main Results

Theorem 3.1: Let $\alpha \neq 0$, be a complex number. Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that $\frac{\alpha zq'(z)}{q(z)}$ is starlike in $\mathbb{E}$. Suppose that

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{\alpha - 1}{\alpha}q(z)\right) > 0, z \in \mathbb{E}. \qquad (6)$$

If $p,\ p(z) \neq 0, z \in \mathbb{E}$, satisfies the differential subordination

$$(1-\alpha)(1-p(z)) + \alpha\frac{zp'(z)}{p(z)} \\ \prec (1-\alpha)(1-q(z)) + \alpha\frac{zq'(z)}{q(z)}, \qquad (7)$$

then $p \prec q$ and q is the best dominant.

Proof: Let us define a function

$$h(z) = (1-\alpha)(1-q(z)) + \alpha\frac{zq'(z)}{q(z)},\ z \in \mathbb{E}. \qquad (8)$$

Differentiate (8) and simplify a little, we get

$$\frac{zh'(z)}{Q_1(z)} = \frac{\alpha - 1}{\alpha}q(z) + 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)},$$

where $Q_1(z) = \dfrac{\alpha z q'(z)}{q(z)}$.

Using (6), we have

$$\Re\left(\frac{zh'(z)}{Q_1(z)}\right) > 0,\ z \in \mathbb{E}.$$

Since Q_1 is starlike, therefore h is close-to-convex and hence univalent in $\mathbb{E}$. The subordination in (7) is, therefore, well-defined in $\mathbb{E}$.

We need to show that $p \prec q$. Suppose to the contrary that $p \not\prec q$ in $\mathbb{E}$. Then by Lemma 2.4, there exist points $z_0 \in \mathbb{E}$ and $\zeta_0 \in \partial\mathbb{E}$ such that $p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0),\ m \geq 1$. Then

$$(1-\alpha)(1-p(z_0)) + \alpha\frac{z_0 p'(z_0)}{p(z_0)} = (1-\alpha)(1-q(\zeta_0)) + \alpha\frac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}. \quad (9)$$

Consider a function

$$L(z,t) = (1-\alpha)(1-q(z)) + \alpha(1+t)\frac{zq'(z)}{q(z)},\ z \in \mathbb{E}. \quad (10)$$

The function $L(z,t)$ is analytic in $\mathbb{E}$ for all $t \geq 0$ and is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{E}$. Now, a little calculation shows

$$\frac{\partial L(z;t)}{\partial z} = \alpha\frac{q'(z)}{q(z)}\left[\frac{\alpha-1}{\alpha}q(z) + (1+t)\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right)\right].$$

Therefore,

$$a_1(t) = \frac{\partial L(0,t)}{\partial z} = \alpha\frac{q'(0)}{q(0)}\left[\frac{\alpha-1}{\alpha}q(0) + 1 + t\right],$$

as q is univalent in $\mathbb{E}$, so, $q'(0) \neq 0$ and $\alpha \neq 0$.

Now, using (6) for $z = 0$, we conclude that $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$.

Further, a simple calculation yields

$$z\frac{\partial L/\partial z}{\partial L/\partial t} = \frac{\alpha-1}{\alpha}q(z) + (1+t)\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right).$$

Since Q_1 is starlike, therefore, in view of the condition (6), we obtain

$$\Re\left(z\frac{\partial L/\partial z}{\partial L/\partial t}\right) > 0,\ z \in \mathbb{E}.$$

Hence, in view of Lemma 2.3, $L(z,t)$ is a subordination chain. Therefore, $L(z,t_1) \prec L(z,t_2)$ for $0 \leq t_1 \leq t_2$. From (10), we have $L(z,0) = h(z)$, thus we deduce that $L(\zeta_0,t) \notin h(\mathbb{E})$ for $|\zeta_0| = 1$ and $t \geq 0$. In view of (9) and (10), we can write

$$(1-\alpha)(1-p(z_0)) + \alpha\frac{z_0 p'(z_0)}{p(z_0)} = L(\zeta_0, m-1) \notin h(\mathbb{E}),$$

where $z_0 \in \mathbb{E}$, $|\zeta_0| = 1$ and $m \geq 1$ which is a contradiction to (7). Hence, $p \prec q$. This completes the proof of the theorem. ■

Theorem 3.2: Let $\alpha(\neq 0,1)$, be a complex number. Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$. Suppose that

$$\Re\left(\frac{\alpha-1}{\alpha}q(z) + t\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right)\right) > 0, \quad (11)$$

for all $z \in \mathbb{E},\ t \geq 0$.

Let $p \in \mathcal{H}[q(0),1] \cap Q$, with $p(z) \neq 0, z \in \mathbb{E}$, be such that $(1-\alpha)(1-p(z)) + \alpha\dfrac{zp'(z)}{p(z)}$ is univalent in $\mathbb{E}$. If p satisfies the differential superordination

$$h(z) = (1-\alpha)(1-q(z)) + \alpha\frac{zq'(z)}{q(z)} \prec (1-\alpha)(1-p(z)) + \alpha\frac{zp'(z)}{p(z)}, \quad (12)$$

then $q \prec p$ and q is the best subordinant.

Proof: Let us define ϕ as follows

$$\phi(p(z), zp'(z)) = (1-\alpha)(1-p(z)) + \alpha\frac{zp'(z)}{p(z)},\ z \in \mathbb{E}.$$

Therefore, (12), becomes

$$h(z) \prec \phi(p(z), zp'(z)),$$

as $\phi(p(z), zp'(z))$, is univalent in $\mathbb{E}$. The subordination in (12) is, therefore, well-defined in $\mathbb{E}$.

Consider a function

$$L(z,t) = (1-\alpha)(1-q(z)) + \alpha\, t\frac{zq'(z)}{q(z)},\ z \in \mathbb{E}.$$

The function $L(z,t)$ is analytic in $\mathbb{E}$ for all $t \geq 0$ and is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{E}$. Now, a little calculation shows

$$\frac{\partial L(z;t)}{\partial z} = \alpha\frac{q'(z)}{q(z)}\left[\frac{\alpha-1}{\alpha}q(z) + t\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right)\right]$$

and

$$a_1(t) = \frac{\partial L(0,t)}{\partial z} = \alpha\frac{q'(0)}{q(0)}\left[\frac{\alpha-1}{\alpha}q(0) + t\right],$$

as q is univalent in $\mathbb{E}$, so, $q'(0) \neq 0$ and $\alpha \neq 0$.

Now, using (11) for $z = 0$, we conclude that $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$.

By calculating a little, we have

$$z\frac{\partial L/\partial z}{\partial L/\partial t} = \frac{\alpha-1}{\alpha}q(z) + t\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right).$$

In view of the condition (11), we obtain

$$\Re\left(z\frac{\partial L/\partial z}{\partial L/\partial t}\right) > 0,\ z \in \mathbb{E}.$$

Hence, in view of Lemma 2.3, $L(z,t)$ is a subordination chain. Now, the use of Theorem 2.5, completes the proof. ■

On combining Theorem 3.1 and Theorem 3.2, we obtain the following sandwich-type theorem.

Theorem 3.3: Let $\alpha(\neq 0,1)$, be a complex number. Let $q_1,\ q_1(z)\neq 0,\ q_2,\ q_2\neq 0$, be univalent functions in $\mathbb{E}$ with $q_1(0)=q_2(0)$. Set

$$\Psi(q_i(z;t))=\frac{\alpha-1}{\alpha}q_i(z)+t\left(1+\frac{zq_i''(z)}{q_i'(z)}-\frac{zq_i'(z)}{q_i(z)}\right),$$

for $i=1,2$ and

$$\Phi(p(z),zp'(z))=(1-\alpha)(1-p(z))+\alpha\frac{zp'(z)}{p(z)}.$$

Suppose that

$$\Re[\Psi(q_1(z;t))]>0,\ z\in\mathbb{E},\ t\geq 0,$$
$$\Re[\Psi(q_2(z;1))]>0,\ z\in\mathbb{E},$$

and $\dfrac{\alpha zq_2'(z)}{q_2(z)}$, is starlike in $\mathbb{E}$. If $p\in\mathcal{H}[q_1(0),1]\cap Q$, with $p(z)\neq 0, z\in\mathbb{E}$, is such that $\Phi(p(z),zp'(z))$ is univalent in $\mathbb{E}$, then

$$\Phi(q_1(z),zq_1'(z))\prec\Phi(p(z),zp'(z))\prec\Phi(q_2(z),zq_2'(z))$$
$$\Rightarrow q_1(z)\prec p(z)\prec q_2(z).$$

Moreover, q_1 and q_2 are respectively the best subordinant and best dominant.

IV. Applications to Univalent Functions

In this section, we shall restrict ourself to the applications of Theorem 3.1 to univalent functions. However, for superordination case Theorem 3.2 can be applied to obtain parallel results.

On writing $p(z)=f'(z)$ in Theorem 3.1, we obtain the following result.

Theorem 4.1: Let $\alpha\neq 0$, be a complex number. Let $q,\ q(z)\neq 0$, be a univalent function in $\mathbb{E}$ such that $\dfrac{\alpha zq'(z)}{q(z)}$ is starlike in $\mathbb{E}$ and satisfy the condition (6) of Theorem 3.1. If $f\in\mathcal{A},\ f'(z)\neq 0,\ z\in\mathbb{E}$, satisfies the differential subordination

$$(1-\alpha)(1-f'(z))+\alpha\frac{zf''(z)}{f'(z)}\prec(1-\alpha)(1-q(z))+\alpha\frac{zq'(z)}{q(z)},$$

then $f'(z)\prec q(z)$ and q is the best dominant.

On writing $p(z)=\dfrac{zf'(z)}{f(z)}$ in Theorem 3.1, we obtain the following result.

Theorem 4.2: Let $\alpha\neq 0$, be a complex number. Let $q,\ q(z)\neq 0$, be a univalent function in $\mathbb{E}$ such that $\dfrac{\alpha zq'(z)}{q(z)}$ is starlike in $\mathbb{E}$ and satisfy the condition (6) of Theorem 3.1. If $f\in\mathcal{A},\ \dfrac{zf'(z)}{f(z)}\neq 0,\ z\in\mathbb{E}$, satisfies the differential subordination

$$1+\alpha\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\prec(1-\alpha)(1-q(z))+\alpha\frac{zq'(z)}{q(z)},$$

then $\dfrac{zf'(z)}{f(z)}\prec q(z)$ and q is the best dominant.

Remark 4.3: When we select the dominant $q(z)=\dfrac{1+z}{1-z},\ z\in\mathbb{E}$, then

$$Q_1(z)=\frac{\alpha zq'(z)}{q(z)}=\frac{2\alpha z}{1-z^2},$$

and

$$\frac{zQ_1'(z)}{Q_1(z)}=\frac{1+z^2}{1-z^2}.$$

Therefore, we have

$$\Re\left(\frac{zQ_1'(z)}{Q_1(z)}\right)>0,\ z\in\mathbb{E},$$

and hence Q_1 is starlike.

We also have

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{\alpha-1}{\alpha}q(z)=\frac{1+z^2}{1-z^2}+\frac{\alpha-1}{\alpha}\frac{1+z}{1-z}.$$

Thus, for real number $\alpha\in(-\infty,0)\cup(1/2,\infty)$, we obtain

$$\Re\left[1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{\alpha-1}{\alpha}q(z)\right]>0, z\in\mathbb{E}.$$

Therefore, $q(z)=\dfrac{1+z}{1-z},\ z\in\mathbb{E}$, satisfies the conditions of Theorem 4.1 and Theorem 4.2 and so we get.

Corollary 4.4: Let $\alpha\in(-\infty,0)\cup(1/2,\infty)$, be a real number. Let $f\in\mathcal{A},\ f'(z)\neq 0,\ z\in\mathbb{E}$, satisfy the differential subordination

$$(1-\alpha)(1-f'(z))+\alpha\frac{zf''(z)}{f'(z)}\prec 2(\alpha-1)\frac{z}{1-z}+2\alpha\frac{z}{1-z^2}=F(z),$$

then $\Re(f'(z))>0,\ z\in\mathbb{E}$. So f is close-to-convex and hence univalent in $\mathbb{E}$.

It is easy to check that for real number $\alpha\in(-\infty,0)\cup[2/3,\infty)$, F is the conformal mapping of the unit disc $\mathbb{E}$ ($F(0)=0$) and

$$F(\mathbb{E})=\mathbb{C}\setminus\left\{w\in\mathbb{C}:\Re(w)=1-\alpha,\ |\Im(w)|\geq\sqrt{3\alpha^2-2\alpha}\right\}. \quad (13)$$

Hence we obtain the following result.

Corollary 4.5: Let α, be a real number such that $\alpha\in(-\infty,0)\cup[2/3,\infty)$. Let $f\in\mathcal{A},\ f'(z)\neq 0,\ z\in\mathbb{E}$, satisfy the condition

$$(1-\alpha)(1-f'(z))+\alpha\frac{zf''(z)}{f'(z)}\prec F(z),$$

then f is close-to-convex and hence univalent in $\mathbb{E}$, where $F(\mathbb{E})$ is given by (13).

In view of Corollary 4.4, we obtain the following result.

Corollary 4.6: Let α, be a real number such that $\alpha\in(-\infty,0)\cup(1/2,1)$. Let $f\in\mathcal{A},\ f'(z)\neq 0,\ z\in\mathbb{E}$, satisfy the condition

$$\Re\left((1-\alpha)(1-f'(z))+\alpha\frac{zf''(z)}{f'(z)}\right)<1-\alpha,$$

then f is close-to-convex and hence univalent in $\mathbb{E}$.

From Corollary 4.4, we obtain the following result.

Corollary 4.7: Let α, be a real number such that $1 < \alpha < \infty$. Let $f \in \mathcal{A}$, $f'(z) \neq 0,\ z \in \mathbb{E}$, satisfy the condition

$$\Re\left((1-\alpha)(1-f'(z)) + \alpha\frac{zf''(z)}{f'(z)}\right) > 1-\alpha,$$

then f is close-to-convex and hence univalent in $\mathbb{E}$.

Keeping in view the Remark 4.3 and by taking $q(z) = \dfrac{1+z}{1-z}$ in Theorem 4.2, we obtain the following result.

Corollary 4.8: Let $\alpha \in (-\infty, 0) \cup (1/2, \infty)$, be a real number. Let $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfy the differential subordination

$$1 + \alpha\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec 2(\alpha-1)\frac{z}{1-z} + 2\alpha\frac{z}{1-z^2} = F(z),$$

then $f \in S^*$.

From Corollary 4.8, we conclude the following result.

Corollary 4.9: Let α, be a real number such that $\alpha \in (-\infty, 0) \cup [2/3, \infty)$. Let $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfy the condition

$$1 + \alpha\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec F(z),$$

then $f \in S^*$, where $F(\mathbb{E})$ is as given in (13).

Remark 4.10: As mentioned in Section 1, Corollary 4.9, offers the correct form of the result stated in Theorem 1.2. For positive real values of α, both the results coincide. But for negative real values of α, the results differ due to different images of the unit disc $\mathbb{E}$ under superordinate functions F and G. We show the comparison below pictorially.

For $\alpha = 2$, the image of the unit disc $\mathbb{E}$ under superordinate functions F and G, is the entire complex plane except two slits $\{\Re(w) = -1,\ |\Im(w)| > 2\sqrt{2}\}$ parallel to imaginary axis. Figure 1, shows that our result coincides with that of Obradovic, et al. [5].

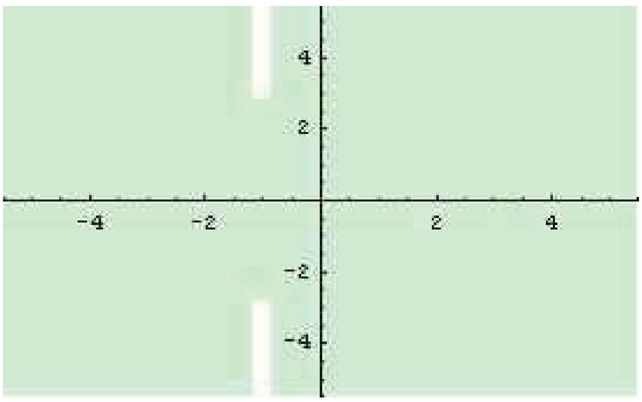

Fig. 1 ($\alpha = 2$)

For $\alpha = -2$, we plot the image of the unit disc $\mathbb{E}$ under superordinate function F (Figure 2) and $G(\mathbb{E})$ is plotted in Figure 3. Both are different. The slits in Figure 3 are placed at $\Re(w) = -3$, but their correct place is $\Re(w) = 3$ as shown in Figure 2. This justifies our claim as mentioned above.

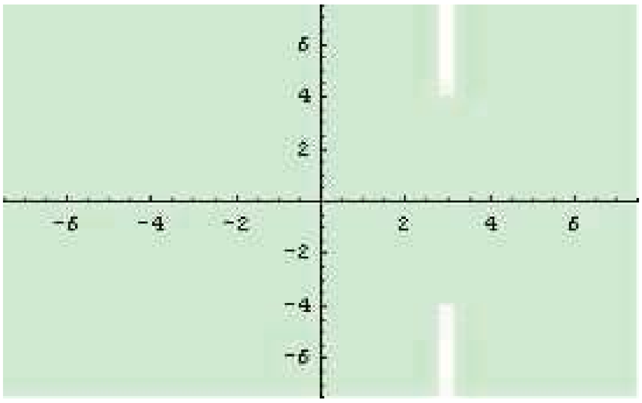

Fig. 2 ($\alpha = -2$)

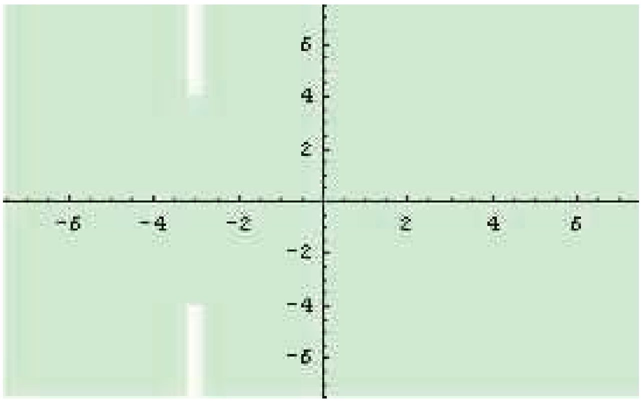

Fig. 3 ($\alpha = -2$)

In view of Corollary 4.8, we obtain the following result that extends the result of Obradovic, et al. [5].

Corollary 4.11: Let α, be a real number such that $\alpha \in (-\infty, 0) \cup (1/2, 1)$. Let $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfy the condition

$$\Re\left(1 + \alpha\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) < 1-\alpha,$$

then $f \in S^*$.

In view of Corollary 4.8, we also obtain the following result.

Corollary 4.12: Let α, be a real number such that $1 < \alpha < \infty$. Let $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfy the condition

$$\Re\left(1 + \alpha\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) > 1-\alpha,$$

then $f \in S^*$.

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