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Constructing Approximate and Exact Solutions for Boussinesq Equations using Homotopy Perturbation Padé Technique

Mohamed M. Mousa, and Aidarkhan Kaltayev

Abstract**—**Based on the homotopy perturbation method (HPM) and Padé approximants (PA), approximate and exact solutions are obtained for cubic Boussinesq and modified Boussinesq equations. The obtained solutions contain solitary waves, rational solutions. HPM is used for analytic treatment to those equations and PA for increasing the convergence region of the HPM analytical solution. The results reveal that the HPM with the enhancement of PA is a very effective, convenient and quite accurate to such types of partial differential equations.

Keywords**—**Homotopy perturbation method, Padé approximants, cubic Boussinesq equation, modified Boussinesq equation.

I. Introduction

THE homotopy perturbation method (HPM) was firstly proposed by He [1–4]. It deforms a difficult problem into simple problems that can easily be solved. The HPM, based on a series approximation, is one among the newly developed analytical methods for strongly nonlinear problems and has been proven successful and efficiency in solving a wide class of nonlinear differential equations, various physics and engineering problems [5–7]. But, the convergence region of the obtained truncated series approximation is limited and, in great, need of enhancements to enlarge convergence region of the approximate solution.

It's well known that Padé approximations (PA) [8] have the advantage of manipulating the polynomial approximation into a rational function of polynomials. This manipulation provides us with more information about the mathematical behavior of the solution. So, the application of PA to the truncated series solution obtained by HPM will be an effective tool to increase the region of convergence and accuracy of the approximate solution even for large values of *t*.

In this paper, we are interested in applying the HPM with Padé technique to obtain approximate analytical and exact solutions of the cubic Boussinesq and modified Boussinesq equations. Comparison of the present solutions is made with the exact ones and excellent agreement is noted.

M. M. Mousa is with the Department of Basic Science, Benha High Institute of Technology, Benha University, 13512, Egypt (corresponding author to provide e-mail: engm_medhat@yahoo.com).

A. Kaltayev is with Department of Mechanics, al-Farabi Kazak National University, 39/47 Masanchi 480012, Almaty, Kazakstan (e-mail: Aidarkhan. Kaltayev@kaznu.kz).

II. Basic Idea of He's Homotopy Perturbation Method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation [1]:

$$A(u)-f(r)=0, \quad r\in\Omega, \tag{1}$$

with the boundary conditions

$$B(u,\partial u/\partial n)=0, \quad r\in\Gamma, \tag{2}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

Generally speaking, the operator A can be divided into two parts which are L and N, where L is linear, but N is nonlinear. Therefore, (1) can be rewritten as follows:

$$L(u)+N(u)-f(r)=0. \tag{3}$$

By the homotopy technique, we construct a homotopy $V(r,p):\Omega\times[0,1]\to\mathbb{R}$ which satisfies:

$$\begin{aligned} H(V,p)=(1-p)\left[L(V)-L(u_0)\right] \\ +p\left[A(V)-f(r)\right]=0, \quad p\in[0,1], \ r\in\Omega, \end{aligned} \tag{4}$$

where $p\in[0,1]$ is an embedding parameter, u_0 is an initial approximation of (1), which satisfies the boundary conditions. Obviously, from (4) we will have:

$$H(V,0)=L(V)-L(u_0)=0, \tag{5}$$

$$H(V,1)=A(V)-f(r)=0, \tag{6}$$

and the changing process of p from zero to unity is just that of $V(r,p)$ from $u_0(r)$ to $u(r)$.

According to the HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solution of (4) can be written as a power series in p:

$$V = V_0 + pV_1 + p^2 V_2 + ... \quad (7)$$

Setting $p = 1$ results in the approximate solution of (1):

$$u = \lim_{p \to 1} V = V_0 + V_1 + V_2 + ... \quad (8)$$

The series in (8) is convergent for most cases, and also the rate of convergent depends on the nonlinear operator $A(V)$ [1]. So we can calculate the terms of $u = \sum_{n=0}^{\infty} V_n$, term by term, otherwise, by computing some terms say k, $u \approx \varphi_k = \sum_{n=0}^{k} V_n$, where, $u = \lim_{k \to \infty} \varphi_k$ an approximation to the solution would be achieved.

III. Applying Padé Approximants to the Truncated Series Solution

Let the rational approximation of a truncated series solution $f(t)$ of order at least $(N + M)$ on $[a, b]$ is the quotient of two polynomials $P_N(t)$ and $Q_M(t)$ of degree N and M, respectively. We use the notation $[N/M]$ to denote the quotient:

$$[N/M] = P_N(t)/Q_M(t) \quad \text{for} \quad a \le t \le b. \quad (9)$$

The goal of Padé method is to make the maximum error as small as possible. The method of Padé requires that $f(t)$ and its derivative be continuous at t=0. Let the polynomials used in (9) are

$$P_N(t) = p_0 + p_1 t + p_2 t^2 + ... + p_N t^N, \quad (10)$$

$$Q_M(t) = 1 + q_1 t + q_2 t^2 + ... + q_M t^M, \quad (11)$$

The polynomials in (10) and (11) are constructed so that $f(t)$ and $[N/M]$ agree at t=0 and their derivatives up to N+M agree at t=0.

Notice that the constant coefficient of Q_M is q_0=1. Hence the rational function $[N/M]$ has N+M+1 unknown coefficient. Assume that $f(t)$ is analytic and has the Maclaurin expansion,

$$f(t) = a_0 + a_1 t + a_2 t^2 + ... + a_k t^k + ... \quad (12)$$

The formal power series,

$$f(t) - [P_N(t)/Q_M(t)] = O\left(t^{N+M+1}\right), \quad (13)$$

determines the coefficients of $P_N(t)$ and $Q_M(t)$.

Multiplying (13) by $Q_M(t)$ gives

$$\left(\sum_{j=0}^{\infty} a_j t^j\right)\left(\sum_{j=0}^{M} q_j t^j\right) - \left(\sum_{j=0}^{N} p_j t^j\right) = \left(\sum_{j=N+M+1}^{\infty} c_j t^j\right). \quad (14)$$

When the left side of (14) multiplied out and the coefficients of the powers of t^k are set equal to zero for k=0, 1, …, N+M, the result is a system of N+M+1 linear equations:

$$\begin{aligned} a_0 - b_0 &= 0, \\ q_1 a_0 + a_1 - b_1 &= 0, \\ q_2 a_0 + q_1 a_1 + a_2 - b_2 &= 0, \\ q_3 a_0 + q_2 a_1 + q_1 a_2 + a_3 - b_3 &= 0, \\ \vdots \qquad & \quad \vdots \\ q_M a_{N-M} + q_{M-1} a_{N-M+1} + ... + a_N - b_N &= 0, \end{aligned} \quad (15)$$

and

$$\begin{aligned} q_M a_{N-M+1} + q_{M-1} a_{N-M+2} + ... + q_1 a_N + a_{N+1} &= 0, \\ q_M a_{N-M+2} + q_{M-1} a_{N-M+3} + ... + q_1 a_{N+1} + a_{N+2} &= 0, \\ \vdots \qquad & \quad \vdots \\ q_M a_N + q_{M-1} a_{N+1} + ... + q_1 a_{N+M-1} + a_{N+M} &= 0. \end{aligned} \quad (16)$$

The M equations in (16) involve only the unknowns q_1, q_2, …, q_M, and must be solved first. Then the equations in (15) are used successively to find p_0, p_1, …, p_N.

To obtain Padé approximants of different orders $[N/M]$, Maple or Mathematica can be efficiently used.

IV. Applications

A. Cubic modified Boussinesq equations

Firstly, consider the following general equation [9, 10],

$$u_{tt} + \alpha u_{xxt} + \beta u_{xxxx} + \gamma \left(u^n\right)_{xx} = 0, \quad (17)$$

where α, β, γ and n are real constants.

This equation is called the high-order modified Boussinesq equation with the damping term u_{xxt}. It appears in several domains of mathematics and physics.

When $\alpha = 0$, $\beta = -1$, $\gamma = 1$ and $n = 3$, (17) becomes the modified Boussinesq equation which was presented in the famous Fermi–Pasta–Ulam problem and is used to investigate the behavior of systems which are primarily linear but a nonlinearity is introduced as a perturbation. It also arises in other physical applications.

Example 1:

Consider the cubic modified Boussinesq equation,

$$u_{tt}+u_{xxt}+\frac{2}{9}u_{xxxx}-\left(u^3\right)_{xx}=0,$$
$$u(x,0)=1+\tanh\left(\frac{3}{2}x\right), \tag{18}$$
$$u_t(x,0)=-3\operatorname{sech}^2\left(\frac{3}{2}x\right).$$

According to (4), a homotopy $V(x,t,p):\Omega\times[0,1]\to\mathbb{R}$ can be constructed as follows:

$$(1-p)\left(V_{tt}-u_{0,tt}\right)+p\left(V_{tt}+V_{xxt}+\frac{2}{9}V_{xxxx}-\left(V^3\right)_{xx}\right)=0, \quad p\in[0,1],\ (x,t)\in\Omega, \tag{19}$$

where $u_0=u_0(x,t)=V_0(x,0)=u(x,0)$ and $u_{0,tt}=\frac{\partial^2 u_0}{\partial t^2}$. One can now try to obtain a solution of (10) in the form of:

$$V(x,t)=V_0(x,t)+p\,V_1(x,t)+p^2\,V_2(x,t)+.... \tag{20}$$

Substituting (11) into (10), and equating the terms with the identical powers of p, yields:

$$\begin{aligned}
p^0 &: V_{0,tt}=0,\\
p^1 &: V_{1,tt}+V_{0,xxt}+\frac{2}{9}V_{0,xxxx}-3\left(V_0\right)^2V_{0,xx}\\
&\quad -6V_0\left(V_{0,x}\right)^2=0,\\
p^2 &: V_{2,tt}+V_{1,xxt}+\frac{2}{9}V_{1,xxxx}-3\left(V_0\right)^2V_{1,xx}-6V_0V_1V_{0,xx}\\
&\quad -6V_1\left(V_{0,x}\right)^2-12V_0V_{0,x}V_{1,x}=0,\\
&\vdots,
\end{aligned} \tag{21}$$

with the following initial conditions:

$$V_i(x,0)=\begin{cases}1+\tanh\left(\frac{3}{2}x\right), & i=0,\\ 0, & i=1,2,...,\end{cases}$$
$$V_{i,t}(x,0)=\begin{cases}-3\operatorname{sech}^2\left(\frac{3}{2}x\right), & i=0,\\ 0, & i=1,2,....\end{cases} \tag{22}$$

Solving the system (12), with the conditions (13), gives the following:

$$\begin{aligned}
V_0(x,t)&=1+\tanh\left(\frac{3}{2}x\right)-3t\operatorname{sech}^2\left(\frac{3}{2}x\right),\\
V_1(x,t)&=-9t^2\tanh\left(\frac{3}{2}x\right)\operatorname{sech}^2\left(\frac{3}{2}x\right)\\
&-\frac{9}{40}t^3\{80\operatorname{sech}^2\left(\frac{3}{2}x\right)-120\operatorname{sech}^4\left(\frac{3}{2}x\right)\\
&+120\tanh\left(\frac{3}{2}x\right)\operatorname{sech}^2\left(\frac{3}{2}x\right)-360\tanh\left(\frac{3}{2}x\right)\\
&.\operatorname{sech}^4\left(\frac{3}{2}x\right)\}-\frac{9}{40}t^4\{450\operatorname{sech}^6\left(\frac{3}{2}x\right)\\
&-360\operatorname{sech}^4\left(\frac{3}{2}x\right)+675\tanh\left(\frac{3}{2}x\right)\operatorname{sech}^6\left(\frac{3}{2}x\right)\\
&-360\tanh\left(\frac{3}{2}x\right)\operatorname{sech}^4\left(\frac{3}{2}x\right)\}\\
&-\frac{9}{40}t^5\{486\operatorname{sech}^6\left(\frac{3}{2}x\right)-567\operatorname{sech}^8\},\\
&\vdots.
\end{aligned} \tag{23}$$

In this manner the other components can be easily obtained. For this example, we continued solving (21) for V_n, n=0,1,… until V_2 and hence obtained the two-term approximation φ_2= V_0+ V_1+ V_2.

Applying PA with respect to t to the obtained result φ_2, the rational approximation [N/M] can be obtained such that $N+M \leq 9$ (highest power of the variable t in φ_2) with the aid of Maple or Mathematica software.

In order to evaluate the reliability and accuracy of the HPM with the enhancement of PA, here we mention that (18) has an exact solution,

$$u(x,t)=1+\tanh\left(\frac{3}{2}x-3t\right). \tag{24}$$

TABLE I
RESULTS OBTAINED FOR THE ABSOLUTE ERROR BETWEEN THE EXACT SOLUTION AND φ_2 OBTAINED BY APPLICATION OF HPM

$t_i \backslash x_i$	0.0	0.5	1.0	1.5	2.0
0.1	0.2372496E–3	0.1051763 E–2	0.826206 E–3	0.47619 E–4	0.35346 E–4
0.3	0.61387796 E–2	0.1028504349	0.38668827 E–1	0.2691527 E–2	0.631936 E–3
0.5	0.282770929 E–1	0.2138252261	0.787924531 E–1	0.58406569 E–1	0.8879789 E–1
0.7	3.585856474	3.486293464	0.6280186210	0.3874946876	0.76612254
0.9	43.72139536	29.68321428	4.835834768	1.603806045	1.2539338915
1.1	292.5846898	144.0809202	18.59432219	4.466162969	1.361639355

TABLE II
RESULTS OBTAINED FOR THE ABSOLUTE ERROR BETWEEN THE EXACT SOLUTION AND THE ONE OBTAINED BY APPLICATION OF A PA [2/6] TO HPM SOLUTION φ_2

$t_i \backslash x_i$	0.0	0.5	1.0	1.5	2.0
0.1	0.2434446 E–3	0.1058608 E–2	0.825470 E–3	0.47597 E–4	0.35344 E–4
0.3	0.65226467 E–2	0.547010153 E–1	0.35696911 E–1	0.794166 E–3	0.1181480 E–2
0.5	0.422027631 E–2	0.2137442037	0.638468134 E–1	0.18063867 E–2	0.5222480 E–2
0.7	0.157883936 E–2	0.3489404687	0.145551108 E–1	0.6709987 E–2	0.27240026 E–1
0.9	0.323811305 E–2	0.6899567854	0.697367942 E–1	0.848258196 E–1	0.2258921 E–2
1.1	0.27638871 E–2	0.4006148959	0.697436579 E–1	0.1355118611 E–1	0.13442929 E–1

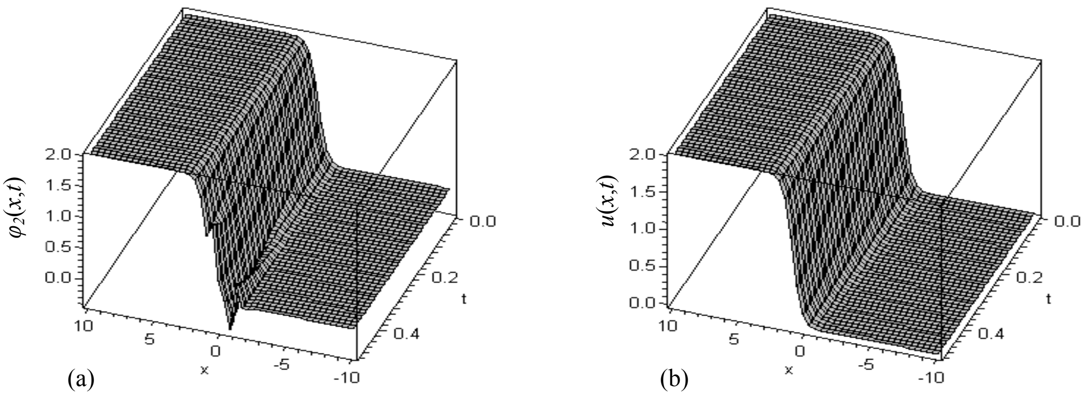

Fig. 1 The surfaces generated from the HPM result $\varphi_2(x,t)$, shown in (a), in comparison with the exact solution (24), shown in (b), for the cubic modified Boussinesq equation (18) in the intervals $0 \le t \le 0.5$ and $-10 \le x \le 10$

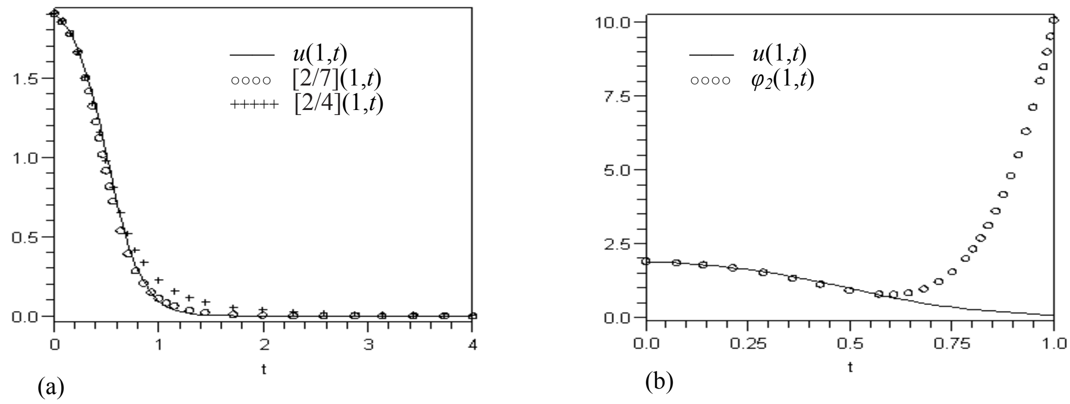

Fig. 2 Results for x =1, obtained for the exact solution (solid line) and the ones obtained by applying PA [2/4] and [2/7] to φ_2, shown in (a), in comparison with HPM result φ_2, shown in (b), for cubic modified Boussinesq equation (18)

The behavior of the solutions φ_2 and the exact solution illustrated in Figs. 1(a), 1(b), 2(b) and 3(b) shows that the HPM solution φ_2 is an accurate approximation only in the interval $0 \le t \le 0.5$ but diverges for values of $t > 0.5$. Also, we can conclude the same result from the numerical results of the absolute error between the exact solution and φ_2 shown in Table I. As well, Figs. 2(b) and 3(b) show the deterioration in the HPM solution φ_2 beyond the interval of convergence.

So, PA would be needed to increase the convergence domain region.

It's seen, from Table II, Figs. 2(a) and 3(a), that the application of PA, for example [2/4], [2/6] and [2/7], to the truncated series solution φ_2 greatly improves the results of the HPM and increases approximate solution accuracy and convergence domain region.

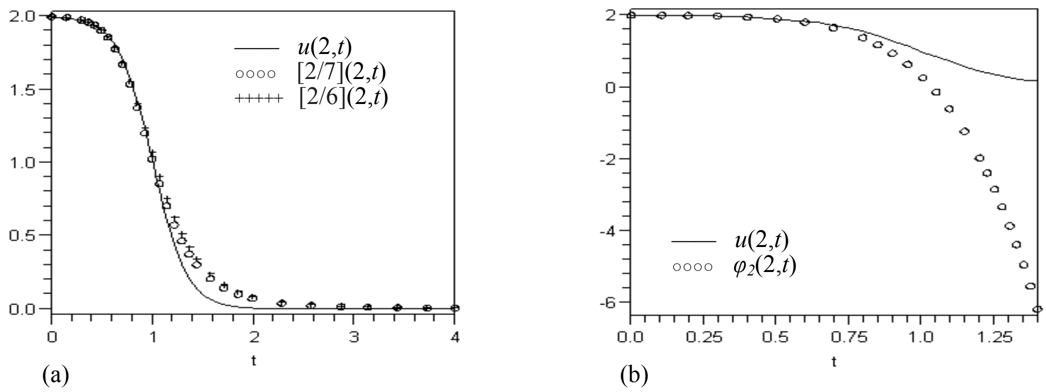

Fig. 3 Results for $x = 2$, obtained for the exact solution (solid line) and the ones obtained by applying PA [2/6] and [2/7] to φ_2, shown in (a), in comparison with HPM result φ_2, shown in (b), for cubic modified Boussinesq equation (18)

Example 2:

Secondly, consider the following modified Boussinesq equation,

$$\begin{aligned} &u_{tt} - u_{xxxx} - \left(u^3\right)_{xx} = 0, \\ &u(x,0) = \sqrt{2}\,\mathrm{sech}(x), \\ &u_t(x,0) = \sqrt{2}\,\mathrm{sech}(x)\tanh(x). \end{aligned} \tag{25}$$

According to (4), we can construct the homotopy $V(x,t,p): \Omega \times [0,1] \to \mathbb{R}$ which satisfies:

$$\begin{aligned} &(1-p)\left(V_{tt} - u_{0,tt}\right) + p\left(V_{tt} - V_{xxxx} - \left(V^3\right)_{xx}\right) = 0, \\ &p \in [0,1], \quad (x,t) \in \Omega. \end{aligned} \tag{26}$$

Suppose the solution of Eq. (26) has the following form,

$$V(x,t) = V_0(x,t) + p\,V_1(x,t) + p^2\,V_2(x,t) + \ldots. \tag{27}$$

Substituting (27) into (26), and equating the terms with the identical powers of p, yields:

$$\begin{aligned} p^0 &: \; V_{0,tt} = 0, \\ p^1 &: \; V_{1,tt} - V_{0,xxxx} - 3(V_0)^2 V_{0,xx} - 6V_0\left(V_{0,x}\right)^2 = 0, \\ p^2 &: \; V_{2,tt} - V_{1,xxxx} - 3(V_0)^2 V_{1,xx} - 6V_0 V_1 V_{0,xx} \\ &\quad - 6V_1\left(V_{0,x}\right)^2 - 12V_0 V_{0,x} V_{1,x} = 0, \\ &\vdots, \end{aligned} \tag{28}$$

with the following initial conditions:

$$\begin{aligned} V_i(x,0) &= \begin{cases} \sqrt{2}\,\mathrm{sech}(x), & i = 0, \\ 0, & i = 1, 2, \ldots, \end{cases} \\ V_{i,t}(x,0) &= \begin{cases} \sqrt{2}\,\mathrm{sech}(x)\tanh(x), & i = 0, \\ 0, & i = 1, 2, \ldots. \end{cases} \end{aligned} \tag{29}$$

The solutions of (28), with the conditions (29), are

$$V_0(x,t)=\sqrt{2}\{\operatorname{sech}(x)+t\operatorname{sech}(x)\tanh(x)\},$$

$$V_1(x,t)=\frac{\sqrt{2}}{30}t^2\{15\operatorname{sech}(x)-30\operatorname{sech}^3(x)\}$$

$$+\frac{\sqrt{2}}{30}t^3\{5\tanh(x)\operatorname{sech}(x)-30\tanh(x)\operatorname{sech}^3(x)\}$$

$$+\frac{\sqrt{2}}{30}t^4\{135\operatorname{sech}^3(x)-555\operatorname{sech}^5(x)+450\operatorname{sech}^7(x)\} \quad (30)$$

$$+\frac{\sqrt{2}}{30}t^5\{27\tanh(x)\operatorname{sech}^3(x)-135\tanh(x)\operatorname{sech}^5(x)$$

$$+126\tanh(x)\operatorname{sech}^7(x)\},$$

$$\vdots$$

For this example, we continued solving (28) for V_n, n=0,1,… until V_3 and therefore obtained the three-term approximation $\varphi_3= V_0+ V_1+ V_2+ V_3$.

Applying Padé technique with respect to t to the HPM result φ_3, the rational approximation [N/M] can be obtained for different values of N and M.

In order to evaluate the dependability and accuracy of the HPM with the enhancement of PA, here we mention that the modified Boussinesq equation (25) has the following solitary wave solution,

$$u(x,t)=\sqrt{2}\operatorname{sech}(x-t). \quad (31)$$

TABLE III
RESULTS OBTAINED FOR THE ABSOLUTE ERROR BETWEEN THE EXACT SOLUTION AND φ_3 OBTAINED BY APPLICATION OF HPM

$t_i \backslash x_i$	0.0	0.5	1.0	1.5	2.0
0.1	0.266098423 E–4	0.138451507 E–6	0.159480863 E–5	0.139724299 E–6	0.82024387 E–8
0.3	0.1752731841	0.777884634 E–3	0.105482384 E–1	0.936840824 E–3	0.54172864 E–4
0.5	10.51718692	0.183009622 E–1	0.6360342725	0.573398506 E–1	0.30817877 E–2
0.7	157.0258159	0.4431236549	9.546672214	0.8735787862	0.42651761 E–1
0.9	1190.626864	10.87148035	72.78198591	6.758549611	0.2918896002
1.1	6041.748206	102.9493967	371.2765615	34.97408851	1.293037367

TABLE IV
RESULTS OBTAINED FOR THE ABSOLUTE ERROR BETWEEN THE EXACT SOLUTION AND THE ONE OBTAINED BY APPLICATION OF A PA [2/4] TO HPM SOLUTION φ_3

$t_i \backslash x_i$	0.0	0.5	1.0	1.5	2.0
0.1	0.0	0.295570634 E–7	0.233345238 E–7	0.427092496 E–7	0.313955411 E–7
0.3	0.183847763 E–8	0.280014285 E–7	0.724077344 E–7	0.394565584 E–7	0.360624458 E–7
0.5	0.948937300 E–7	0.818829652 E–6	0.125242753 E–5	0.616031427 E–6	0.194171522 E–6
0.7	0.114508872 E–5	0.778255865 E–5	0.141702785 E–4	0.954580012 E–5	0.330233009 E–5
0.9	0.668569461 E–5	0.378536887 E–4	0.854958566 E–4	0.679334455 E–4	0.257007859 E–4
1.1	0.251575865 E–4	0.121429750 E–3	0.336500662 E–3	0.318941362 E–3	0.135276457 E–3

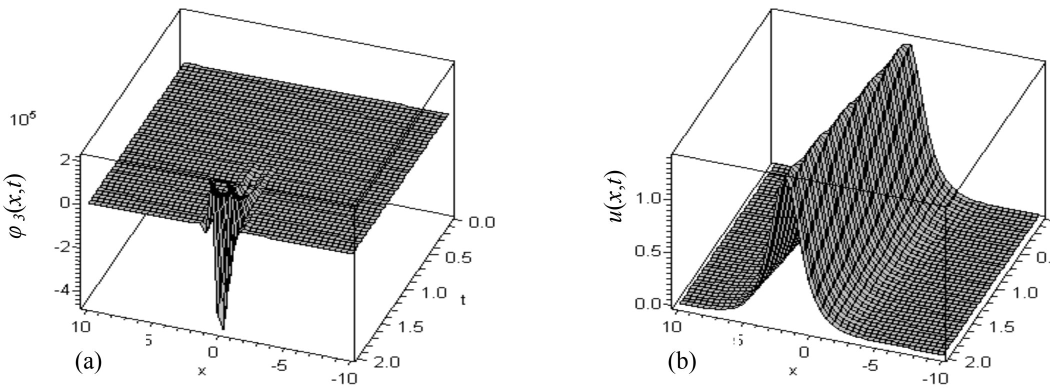

Fig. 4 The surfaces generated from the HPM result $\varphi_3(x,t)$, shown in (a), in comparison with the exact solution (31), shown in (b), for the modified Boussinesq equation (25) in the intervals $0 \le t \le 2$ and $-10 \le x \le 10$

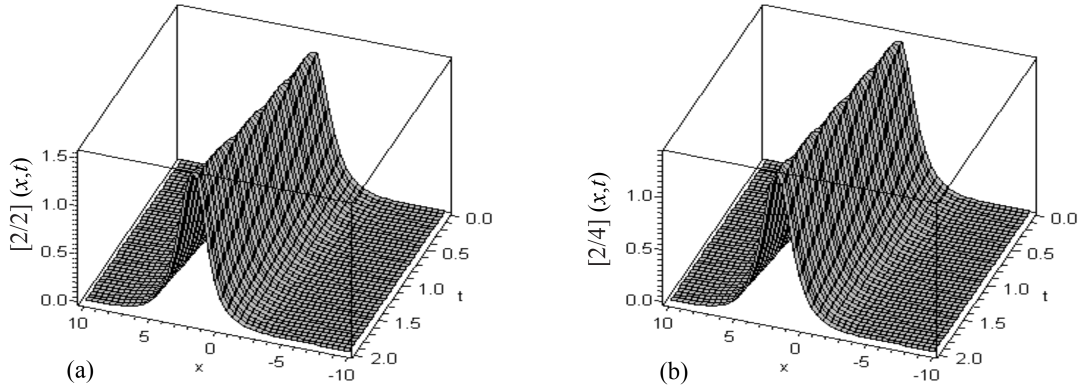

Fig. 5 The surfaces generated from application of PA [2/2] to φ_3, shown in (a), and application of PA [2/4] to φ_3, shown in (b), for the modified Boussinesq equation (25) in the intervals $0 \le t \le 2$ and $-10 \le x \le 10$

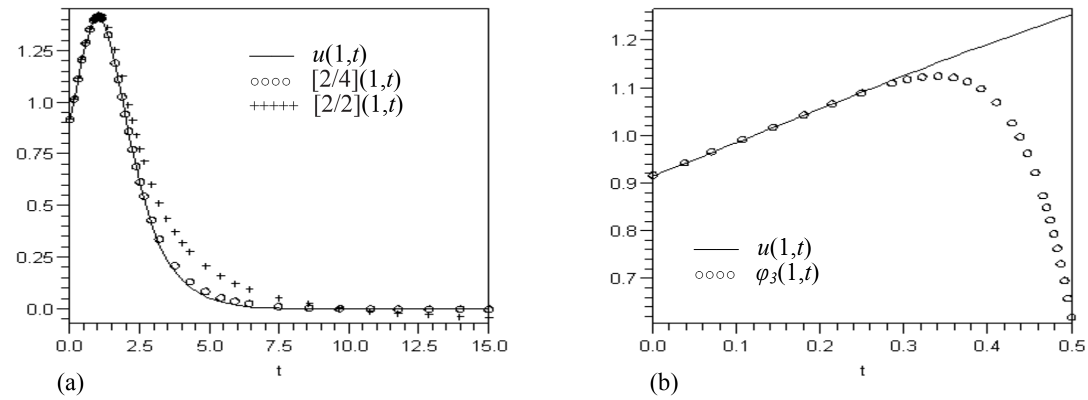

Fig. 6 Results for x =1, obtained for the exact solution (solid line) and the ones obtained by applying PA [2/2] and [2/4] to φ_3, shown in (a), in comparison with HPM result φ_3, shown in (b), for modified Boussinesq equation (25)

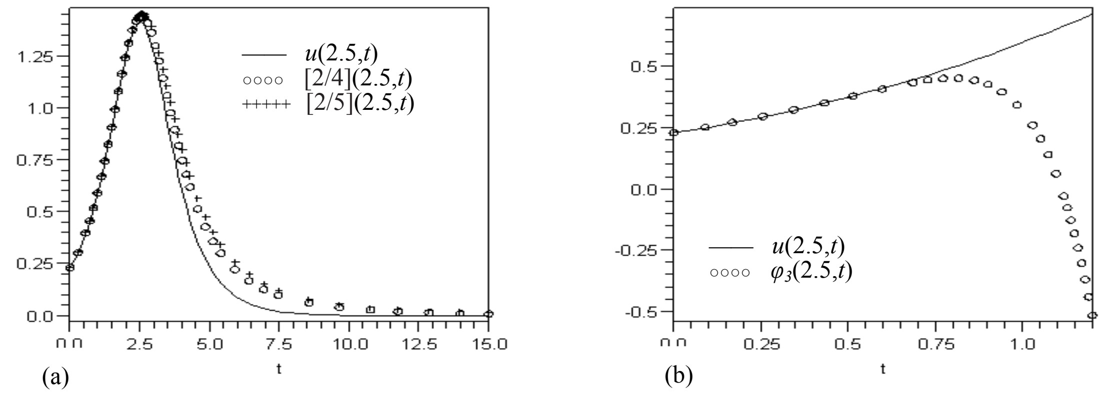

Fig. 7 Results for x =2.5, obtained for the exact solution (solid line) and the ones obtained by applying PA [2/4] and [2/5] to φ_3, shown in (a), in comparison with HPM result φ_3, shown in (b), for modified Boussinesq equation (25)

It's shown, from the behavior of the truncated series solution φ_3 in Fig 4(a), that HPM solution φ_3, in isolation, rapidly diverges for relatively high values of t. Also from Table III, Figs. 6(b) and 7(b), we can conclude that HPM solution is accurate only for a relatively small interval of t but quickly diverges for relatively high values of t. So, PA would be needed to increase the approximate solution accuracy and convergence domain region.

It's shown, in Table IV, Figs. 5, 6(a) and 7(a), that the application of PA to the HPM solution φ_3 really improves the HPM results and increases solution accuracy and convergence domain region.

Now, from examples 1 and 2, we can say that the more accurate solution and improved region of convergence are patent by using PA.

B. Cubic Boussinesq equation

A well-known model of nonlinear dispersive waves which was proposed by Boussinesq is formulated in the form

$$u_{tt} = u_{xx} + 3\left(u^3\right)_{xx} + u_{xxxx}, \qquad L_0 \le x \le L_1. \tag{32}$$

The Boussinesq equation (32) describes motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice.

Now, we apply the homotopy perturbation padé technique to the following cubic Boussinesq equation that considered by wazwaz [11]

$$u_{tt} - u_{xx} - u_{xxxx} + 2\left(u^3\right)_{xx} = 0, \quad u(x,0) = \frac{1}{x}, \quad u_t(x,0) = -\frac{1}{x^2}. \tag{33}$$

The homotopy of (33) $V(x,t,p): \Omega \times [0,1] \to \mathbb{R}$ satisfies

$$(1-p)\left(V_{tt} - u_{0,tt}\right) + p\left(V_{tt} - V_{xx} - V_{xxxx} + 2\left(V^3\right)_{xx}\right) = 0, \quad p \in [0,1], \quad (x,t) \in \Omega. \tag{34}$$

Suppose the solution of (34) to be in the following form,

$$V(x,t) = V_0(x,t) + p\,V_1(x,t) + p^2\,V_2(x,t) + \ldots. \tag{35}$$

Substituting (35) into (34), and equating the terms with the identical powers of p, yields:

$$\begin{aligned} p^0 &: \ V_{0,tt} = 0, \frac{\delta y}{\delta x} \\ p^1 &: \ V_{1,tt} - V_{0,xx} - V_{0,xxxx} + 6\left(V_0\right)^2 V_{0,xx} + 12V_0\left(V_{0,x}\right)^2 = 0, \\ p^2 &: \ V_{2,tt} - V_{1,xx} - V_{1,xxxx} + 6\left(V_0\right)^2 V_{1,xx} + 12V_0V_1V_{0,xx} + 12V_1\left(V_{0,x}\right)^2 + 24V_0V_{0,x}V_{1,x} = 0, \\ &\vdots, \end{aligned} \tag{36}$$

with the following initial condition

$$V_i(x,0) = \begin{cases} \dfrac{1}{x}, & i = 0, \\ 0, & i = 1, 2, \ldots, \end{cases} \qquad V_{i,t}(x,0) = \begin{cases} -\dfrac{1}{x^2}, & i = 0, \\ 0, & i = 1, 2, \ldots. \end{cases} \tag{37}$$

Solving the system (36), with the initial conditions (37), yields:

$$\begin{aligned} V_0(x,t) &= \frac{1}{x} - \frac{t}{x^2}, \\ V_1(x,t) &= \frac{t^2}{x^3} - \frac{t^3}{x^4} - \underbrace{\frac{15t^4}{x^7} + \frac{21t^5}{5x^8}}_{\text{noise terms}}, \\ V_2(x,t) &= \frac{t^4}{x^5} - \frac{t^5}{x^6} + \underbrace{\frac{15t^4}{x^7} - \frac{21t^5}{5x^8} - \frac{308t^6}{5x^9}}_{\text{noise terms}} \\ &\quad \underbrace{-\frac{2250t^6}{x^{11}} + \frac{612t^7}{35x^{10}} + \frac{1782t^7}{7x^{12}} - \frac{11583t^8}{35x^{13}} + \frac{273t^9}{5x^{14}}}_{\text{noise terms}} \\ &\vdots \end{aligned} \tag{38}$$

It's noticed that each V_n contains terms that have similar terms in V_{n+1}, n=1,2,… but with a different sign. These terms called by noise terms and will be self-cancelled in $\varphi_\infty = \lim_{k\to\infty} \sum_{n=0}^{k} V_n$.

In this problem, we continued solving (36) for V_n, n=0,1,… until V_4 and hence obtained the four-term approximation $\varphi_4 = V_0 + V_1 + V_2 + V_3 + V_4$,

$$\varphi_4(x,t) = \frac{1}{x} - \frac{t}{x^2} + \frac{t^2}{x^3} - \frac{t^3}{x^4} + \frac{t^4}{x^5} - \frac{t^5}{x^6} + \frac{t^6}{x^7} - \frac{t^7}{x^8} + \frac{t^8}{x^9} - \frac{t^9}{x^{10}} + \text{noise terms}. \tag{39}$$

We found that the application of the Padé approximants [N/M], with respect to t, to the solution φ_4, for all $N \ge 0$ and $M \ge 1$ such that $N+M \le 9$, gives

$$[N/M](x,t) = \frac{1}{x+t}, \tag{40}$$

which is the exact solution of the cubic Boussinesq equation (33).

It's obvious that utilizing of PA provides self-examination to the exact solution if the differential equation has a rational exact one by using only a few-term approximate solution obtained by HPM.

V. Conclusion

The first conclusion that can be draw from our results is that the He's homotopy perturbation method is an effective tool to deal with nonlinear dispersive equations of Boussinesq and, in isolation, provides an accurate approximation in a relatively small interval of *t*. The second conclusion is that application of the Padé approximants to HPM truncated series solution greatly improves the results, increases accuracy, enlarge convergence region and in some cases gives self-examination to the exact solution. It's worth noting that the HPM with the enhancement of PA is an effective, simple and quite accurate tool for handling and solving nonlinear dispersive equations and other types of nonlinear PDEs. The various applications of the HPM prove that it's an efficient method to handle nonlinear structures. It's predicted that the homotopy perturbation Padé technique will be found widely applications in science and engineering. The computations associated with the examples in this paper were performed using Maple 10.

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