

Fuzzy T-Neighborhood Groups Acting on Sets

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Abstract—In this paper, The T-G-action topology on a set acted on by a fuzzy T-neighborhood (T-neighborhood, for short) group is defined as a final T-neighborhood topology with respect to a set of maps. We mainly prove that this topology is a T-regular T-neighborhood topology.

Keywords—Fuzzy set, Fuzzy topology, Triangular norm, Separation axioms.

I. INTRODUCTION

A T-neighborhood topology on a set can be defined by several method e.g., via closures, interiors, filters, etc. Sometimes a T-neighborhood topology constructed out of given T-neighborhood topologies may be useful. In the classical theory of topological groups, when a topological group G acts on a set X , it confers a topology on X , called the G -action topology on X . In this paper we develop a fuzzy extension of that notion, in the case G is a T-neighborhood group. Variety of useful characterizations of this T-neighborhood topology are considered. We show that the T-G-action topology τ_X^{T-G} coincides with the final T-neighborhood topology τ_f introduced on X by a set of

functions $\left\{ \hat{g} \right\}$;

$$\hat{g} : G \rightarrow X.$$

II. DEFINITION AND PRELIMINARIES

Definition 2.1. [8] A topological group G acts on a non-empty set X , if to each $g \in G$ and each $x \in X$ there corresponds a unique element gx such that

$$g_2(g_1x) = (g_2g_1)x \quad \forall x \in X \text{ and } g_1, g_2 \in G$$

$$ex = x.$$

When G acts on a set X , two families of functions can be defined as follows:

To each $g \in G$, we define $\hat{g} : X \rightarrow X$,

$$\hat{g}(x) = gx.$$

To each $x \in X$, we define $\hat{x} : G \rightarrow X$,

$$\hat{x}(g) = gx.$$

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We will use two important theorems which introduced in [7]. The first gives necessary and sufficient conditions for a group structure and T-neighborhood system to be compatible, and the second gives necessary and sufficient conditions for a filter to be the T-neighborhood filter of e in a T-neighborhood group.

Theorem 2.1 [7] Let $(G, \cdot)$ be a group and β a T-neighborhood base on G . Then $(G, \cdot, t(\beta))$ is a T-neighborhood group if and only if the following are fulfilled:

(a) For every $a \in G$ we have

$$\beta(a) = \{\zeta_a(\mu) \mid \mu \in \beta(e)\}$$

$$(\text{res. } \beta(a) = \{R_a(\mu) \mid \mu \in \beta(e)\} \text{ and}$$

$$\beta(a) = \{\zeta_a(\mu) \mid \mu \in \beta(e)\} \text{ is a T-neighborhood base at } a.$$

(b) For all $\mu \in \beta(e)$ and for all $\varepsilon \in I_0$ there exists

$v \in \beta(e)$ such that $v \cdot \varepsilon \leq \mu^{-1}$, i.e., r is continuous at e .

(c) For all $\mu \in \beta(e)$ and for all $\varepsilon \in I_0$ there exists $v \in \beta(e)$ such that $v \cdot \varepsilon \leq \mu$, i.e., m is continuous at (e, e) .

(d) For all $\mu \in \beta(e)$, for all $\varepsilon \in I_0$ and for all $x \in G$ there exist $v \in \beta(e)$ such that $I_x \cdot v \cdot I_x^{-1} \cdot \varepsilon \leq \mu$, i.e., int_x is continuous at e .

Where $\zeta_x : G \rightarrow G : z \mapsto xz$ (resp. $R_x : G \rightarrow G : z \mapsto zx$) is the left (resp. right) translation.

Theorem 2.2 [7] Let $(G, \cdot)$ be a group and $\mathfrak{F}$ a family of fuzzy subset of G such that the following hold:

(a) $\mathfrak{F}$ is a filterbasis, such that $\mu(e) = 1$ for all $\mu \in \mathfrak{F}$.

(b) For all $\mu \in \mathfrak{F}$ and for all $\varepsilon \in I_0$ there exists $v \in \mathfrak{F}$ such that $v \cdot \varepsilon \leq \mu^{-1}$.

(c) For all $\mu \in \mathfrak{F}$ and for all $\varepsilon \in I_0$ there exists $v \in \mathfrak{F}$ such that $v \cdot \varepsilon \leq \mu$.

(d) For all $\mu \in \mathfrak{F}$, for all $\varepsilon \in I_0$ and for all $x \in G$ there exists $v \in \mathfrak{F}$ such that $I_x \cdot v \cdot I_x^{-1} \cdot \varepsilon \leq \mu$.

Then there exists a unique T-neighborhood system β such that $\mathfrak{F}$ is T-neighborhood basis for the T-neighborhood system at e , $\beta(e)$ and β is compatible with the group structure. This T-neighborhood system is given by

$$\beta(x) = \{I_x \cdot \mu \mid \mu \in \mathfrak{F}\}^{-1} = \{\mu \cdot I_x \mid \mu \in \mathfrak{F}\}^{-1}, x \in G.$$

III. T-NEIGHBORHOOD TOPOLOGIES INDUCED BY T-NEIGHBORHOOD GROUP ACTIONS ON SET

Definition 3.1. Let $(G, \cdot)$ be a group acting on a set X , then for all $\Gamma \in I^G$, $\mu \in I^X$, $g \in G$ and $x \in X$ we define for all $y \in X$

$$\Gamma\mu(y) = \sup \{\Gamma(g)T\mu(x) : (g, x) \in G \times X \text{ and } gx = y\} \quad (1)$$

Proposition 3.1. Let $(G, \cdot)$ be a group acting on a set X and $\Psi, \Gamma \in I^G, \mu \in I^X$. Then

$$(a) \Psi(\Gamma\mu) \leq (\Psi\Gamma)\mu \quad \text{In particular} \\ \Psi(\Gamma\mu)(y) \leq (\Psi\Gamma)\mu(y)$$

$$(b) \Gamma 1_M = \bigvee_{x \in M} \Gamma 1_x$$

$$(c) \Gamma 1_M(y) = \sup \{\Gamma(g): g \in G \text{ and } g^{-1}y \in M\}$$

$$(d) \Gamma 1_x(y) = \sup \{\Gamma(g): g \in G \text{ and } gx = y\}$$

$$(e) I_g\mu(y) = \sup \{\mu(x): x \in X \text{ and } gx = y\} \\ = \mu(g^{-1}y)$$

Proof: (b)-(e) follow immediately from Definition 3.1.

(a) For any $y \in X$:

$$\begin{aligned} \Psi(\Gamma\mu)(y) &= \sup \{\Psi(g) T \Gamma\mu(x): (g, x) \in G \times X, gx = y\} \\ &= \sup \{\Psi(g) T \sup \{\Gamma(h) T \mu(z): hz = x\}: gx = y\} \\ &= \sup \{\Psi(g) T \Gamma(h) T \mu(z): ghz = y\} \\ &= \sup \{(\Psi \circ_T \Gamma)\mu(y) = \sup \{(\Psi \circ_T \Gamma)(k) T \mu(z): kz = y\} \\ &= \sup \{\sup \{\Psi(g) T \Gamma(h): gh = k\} T \mu(z): \\ &\quad kz = y\} \\ &= \sup \{\Psi(g) T \Gamma(h) T \mu(z): (g, h, z) \\ &\quad \in G \times G \times X \text{ and } ghz = y\} \end{aligned}$$

Hence $\Psi(\Gamma\mu)(y) = (\Psi \circ_T \Gamma)\mu(y) \leq (\Psi\Gamma)\mu(y)$.

If both Γ, μ are crisp, then $\Gamma\mu$ is also crisp and is given by

$$\Gamma\mu = \{gx: g \in \Gamma \text{ and } x \in \mu\}.$$

Note that $\Gamma\mu, \Gamma 1_x, I_g\mu \in I^X$ and $\Gamma 1_x(y) = 0$ if $y \notin$ orbit of x .

Theorem 3.1. Let G be a T-neighborhood group acting on a set X , and let $\mathfrak{R}$ be a fundamental system of G at e . For each $x \in X$, let $\beta_x = \{\Gamma 1_x: \Gamma \in \mathfrak{R}\} \in I^X$. Then $\{\beta_x\}_{x \in X}$ is a T-neighborhood basis on X . The resulting T-neighborhood space is denoted by τ_X^{T-G} . Its fuzzy closure operator $\bar{\cdot}: I^X \rightarrow I^X$ is given by: For all $\eta \in I^X, x \in X$:

$$\bar{\eta}(x) = \inf_{\Gamma \in \mathfrak{R}} \sup_{g \in G} \Gamma(g) T \eta(gx) \quad (2)$$

Proof. First, we verify that $\{\beta_x\}_{x \in X}$ is a T-neighborhood basis in X . Let $x \in X, \Gamma,$

$$\Psi \in \mathfrak{R}, \mu = \Gamma 1_x \in \beta_x, \lambda = \Psi 1_x \in \beta_x$$

$$(i) \mu(x) = \Gamma 1_x(x) = \sup \{\Gamma(g): g \in G \text{ and } gx = x\} \\ \geq \Gamma(e) = 1 \quad (\text{Because } ex = x).$$

(ii) There exists $A \in \mathfrak{R}: \Gamma \wedge \Psi \geq A$. Hence

$$\mu \wedge \lambda = \Gamma 1_x \wedge \Psi 1_x \geq A 1_x,$$

which is in β_x .

(iii) T-kernel condition:

Recall that $\{\mathfrak{R} 1_g\}_{g \in G}$ is a T-neighborhood basis of the T-neighborhood group G Theorem 2.2. Let, as before, $\mu = \Gamma 1_x \in \beta_x$. By the T-kernel condition for

$\Gamma \in \mathfrak{R}$, for all $\varepsilon \in I_0$ there exists a family $\{\Gamma_g I_g \in \mathfrak{R}\}_{g \in G}$ such that for all $g, k \in G$

$$\Gamma_e(k) T (\Gamma_k I_k)(g) \leq \Gamma(g) + \varepsilon. \quad (3)$$

We take $v_x = \Gamma_e I_x$. For each $y \in X$, if $y \notin$ orbit of x , take for v_y any element of $\beta_y = \mathfrak{R}_y$.

If $y \in$ orbit of x , choose some $h \in G$ such that $y = hx$, and

$$\delta + \Gamma_e(h) \geq \sup \{\Gamma_e(k): kx = y\} \quad (4)$$

where $\delta \in I_0$ is a real number that satisfies

$$(b + \delta) T (c + \delta) \leq (b T c) + \varepsilon$$

for all $b, c \in I$. Such δ exists by the uniform continuity of T . Take $v_y = \Gamma_h I_y \in \beta_y$. Then, if $y \notin$ orbit of x , we find for all $z \in X$ that

$$2\varepsilon + \mu(z) \geq v_x(y) T v_y(z)$$

because then $v_x(y) = (\Gamma_e I_x)(y) = 0$. And when $y \in$ orbit of x , we find for all $z \in X$:

$$2\varepsilon + \mu(z) = 2\varepsilon + (\Gamma 1_x)(z)$$

$$= \varepsilon + \sup \{\varepsilon + \Gamma(g): gx = z\}$$

$$\geq \varepsilon + \sup \{\Gamma_e(h) T (\Gamma_h I_h)(g): gx = z\} \text{ by (3)}$$

$$\geq (\Gamma_e(h) + \delta) T \sup \{(\Gamma_h I_h)(g): gx = z\}$$

$$\geq \sup \{\Gamma_e(k): kx = y\} T \sup \{(\Gamma_h I_h)(gh^{-1}): (gh^{-1})(hx) = z\}$$

by (4) Since $hx = y$, then

$$2\varepsilon + \mu(z) \geq (\Gamma_e I_x)(y) T \sup \{(\Gamma_h I_h)(t): ty = z\}$$

$$= (\Gamma_e I_x)(y) T (\Gamma_h I_y)(z)$$

$$= v_x(y) T v_y(z).$$

Thus, the kernel condition holds for $\mu \in \beta_x$ in both cases of y . Finally, for all $\eta \in I^X$

$$\bar{\eta}(x) = \inf_{\mu \in \beta_x} \sup_{y \in X} \mu(y) T \eta(y)$$

$$= \inf_{\Gamma \in \mathfrak{R}} \sup_{y \in X} \eta(y) T (\Gamma 1_x)(y)$$

$$= \inf_{\Gamma \in \mathfrak{R}} \sup_{y \in \text{orbit } x} \eta(y) T \sup \{\Gamma(g): g \in G \text{ and } gx = y\}.$$

Because if $y \notin$ orbit x , then $(\Gamma 1_x)(y) = 0$. Thus,

$$\bar{\eta}(x) = \inf_{\Gamma \in \mathfrak{R}} \sup_{g \in G} \eta(gx) T \Gamma(g),$$

Rendering (2).

Proposition 3.2. Let $\Gamma \in I^G, \varphi \subset I^G, g \in G, x \in X$ then

$$(\Gamma I_g) I_x = (\Gamma I_{gx}) \in I^X, \text{ and hence}$$

$$(\varphi \cdot 1_g) 1_x = \varphi 1_{gx} \subset I^X.$$

Proof:

$$((\Gamma I_g) I_x)(y) = \sup \{(\Gamma I_g)(k): k \in G \text{ and } kx = y\}$$

$$= \sup \{\Gamma(kg^{-1}): k \in G \text{ and } kg^{-1}gx = y\}$$

$$= \sup \{\Gamma(t): t \in G \text{ and } tgx = y\}$$

$$= (\Gamma I_{gx})(y).$$

This completes the proof.

Proposition 3.3. For each filterbasis F in I^G and for $x \in X$,

$$\{\Gamma 1_x: \Gamma \in F^\sim\} \subset \{\Psi 1_x: \Psi \in F\}^\sim \subset I^X \quad (5)$$

Proof: Let $\Gamma \in F^\sim$. Then for all $\varepsilon > 0$ there exists $\Gamma_\varepsilon \in F$ such that $\Gamma + \varepsilon \geq \Gamma_\varepsilon$. Then for all $y \in X$ we have

$$\varepsilon + (\Gamma 1_x)(y) = \varepsilon + \sup \{\Gamma(g): gx = y\}$$

$$= \sup \{\varepsilon + \Gamma(g): gx = y\}$$

$$\geq \sup \{ \Gamma_\varepsilon(g): gx = y \} \\ = (\Gamma_\varepsilon I_x)(y).$$

Thus, $\varepsilon + \Gamma I_x \geq \Gamma_\varepsilon I_x \in \{ \Psi I_x: \Psi \in \mathcal{F} \}$. Hence $\Gamma I_x \in \{ \Psi I_x: \Psi \in \mathcal{F} \}^{\sim}$. This proves (5).

Proposition 3.4. The fuzzy closure operator on X defined in (2) does not depend on the particular choice of a fundamental system $\mathfrak{R}$ of e .

Proof: All fundamental systems $\mathfrak{R}$ of G at e have the same saturation $\mathfrak{R}^{\sim}$. Also, for each $x \in X$

$$\beta_x = \{ \Gamma I_x: \Gamma \in \mathfrak{R} \} \\ \subset \{ \Gamma I_x: \Gamma \in \mathfrak{R}^{\sim} \} \\ \subset \{ \Gamma I_x: \Gamma \in \mathfrak{R} \}^{\sim} = \beta_x^{\sim}.$$

As $\{ \beta_x \}$, $\{ \beta_x^{\sim} \}$ induce the same fuzzy closure operator on X , then the fuzzy closure operator defined in (2) is also given by

$$\eta(x) = \inf_{\Gamma \in \mathfrak{R}} \sup_{g \in G} \Gamma(g) \Gamma \eta(gx) \quad (6)$$

Which is independent of the particular choice of a fundamental system $\mathfrak{R}$ of e .

The following definition is well phrased by virtue of Theorem 3.1, and Proposition 3.4;

Definition 3.2. Let G be a T-neighborhood group acting on a set X . A T-G-action-topology on X denoted by τ_X^{T-G} is introduced through its closure operator $\bar{\cdot}$, defined in (2).

Proposition 3.5. Let $\mathfrak{R}$ be a fundamental system at e of G , $\mu \in \mathfrak{R}$. Then

$$I_g \cdot \mathfrak{R} \cdot I_{g^{-1}} \subset \mathfrak{R}^{\sim} \quad (7)$$

Proof: From condition (d) in Theorem 2.1, for all $\varepsilon > 0$ there exists $v_\varepsilon \in \mathfrak{R}$ such that

$$v_\varepsilon - \varepsilon \leq I_g \cdot \mu \cdot I_{g^{-1}}.$$

This proves that $I_g \cdot \mu \cdot I_{g^{-1}} \in \mathfrak{R}^{\sim}$

Notion: In T-G-action topology

(1) We denote the T-neighborhood system at $x \in X$ by $\mathcal{N}(x)$.

(2) Let $\mathfrak{R}$ be the T-neighborhood system of G at e , $x \in X$. We denote $\mathfrak{R} I_x$ by $\zeta(x)$. Recall that $\zeta^{\sim} = \mathcal{N}^*$, i.e $\zeta(x)$ is a T-neighborhood basis at x for this space.

Definition 3.3. Let $(X, \tau, t(\beta))$ be a T-neighborhood space, M be a non-empty set in X . Then $\mu \in I^X$ is said to be a T-neighborhood of M if μ is a T-neighborhood of all points x in M . It follows that the set of all T-neighborhoods of M (called the T-neighborhood system of M) is the set $\bigwedge_{x \in M} \mathcal{N}(x)$.

Proposition 3.6. Let $\Gamma \in I^G$, $g \in G$, $z \in X$ then

$$I_g^{-1}(\Gamma I_z) = (I_g^{-1} \cdot \Gamma) I_z$$

Proof:

$$I_g^{-1}(\Gamma I_z)(y) = (\Gamma I_z)(gy) \\ = \sup \{ \Gamma(h): h \in G, hz = gy \} \\ = \sup \{ \Gamma(gk): k \in G, kz = y \} \\ (I_g^{-1} \cdot \Gamma) I_z(y) = \sup \{ (I_g^{-1} \cdot \Gamma)(k): kz = y \} \\ = \sup \{ \Gamma(gk): k \in G, kz = y \}$$

Then

$$I_g^{-1}(\Gamma I_z) = (I_g^{-1} \cdot \Gamma) I_z$$

Theorem 3.2. Under this T-neighborhood topology the functions $\{ \hat{g} \}$ are homeomorphisms on X .

Proof: Without loss of generality, we take $\mathfrak{R}$ the whole T-neighborhood system at e . Then from Proposition 3.5, $I_g \cdot \mathfrak{R} \cdot I_{g^{-1}} \subset \mathfrak{R}$. Given $x \in X$,

$g \in G$, $\mathfrak{R} I_{gx}$ is a T-neighborhood basis at gx . Let $\mu \in \mathfrak{R} I_{gx}$ we have $\hat{g}^{-1}(\mu)(y) = \mu(gy) = I_{g^{-1}} \mu(y)$, then $\hat{g}^{-1}(\mu) = I_{g^{-1}} \mu \in I_{g^{-1}} \mathfrak{R} I_{gx}$, and from Proposition 3.6

$$I_g^{-1}(\mathfrak{R} I_{gx}) = (I_g^{-1} \cdot \mathfrak{R}) I_{gx} \\ = (I_g^{-1} \cdot \mathfrak{R} \cdot I_g) I_x \text{ by Proposition 3.2} \\ \subset \mathfrak{R}^{\sim} I_x \text{ by Proposition 3.5} \\ \subset \mathcal{N}(x).$$

i.e., $\hat{g}^{-1}(\mu)$ is a T-neighborhood of x . So by Theorem 5.1 in [5] $\hat{g}$ is continuous at x for all x , and hence it is continuous. Since $(\hat{g}^{-1})^{\sim} = (\hat{g})^{-1}$. Then $(\hat{g})^{-1}$ is also continuous. Thus $\hat{g}$ is a homeomorphism.

Proposition 3.7. For any symmetric T-neighborhood Δ of e , and any $M \subset X$; $x, z \in X$

$$(\Delta I_x)(z) = (\Delta I_z)(x) \\ (\Delta I_M)(x) = \sup_{y \in X} I_M(y) \wedge (\Delta I_x)(y).$$

Proposition 3.8. For any subset M of X and any T-neighborhood Γ of e , ΓI_M is a T-neighborhood of M , and

$$(I_M)^{\sim} \leq \Gamma I_M \in I^X. \quad (8)$$

Proof: Since $\Gamma I_M = \bigvee_{x \in M} \Gamma I_x$, then ΓI_M is a T-neighborhood of all points of M , hence ΓI_M is a T-neighborhood of M .

Next, let Γ be a T-neighborhood of e . Then Γ contains a symmetric T-neighborhood Δ of e . For any $x \in X$

$$\begin{aligned} (I_M)^-(x) &= \inf_{\lambda \in \mathcal{C}} \sup_{y \in X} I_M(y) T \lambda(y) \\ &\leq \sup_{y \in X} I_M(y) T \Delta I_x(y) \\ &= \sup_{y \in M} \Delta I_x(y) \\ &= \sup_{y \in M} \Delta I_y(x) \quad \text{by Proposition 3.7} \\ &= (\Delta I_M)(x) \\ &\leq (\Gamma I_M)(x). \text{ This proves (8).} \end{aligned}$$

Proposition 3.9. Let $\mathfrak{R}$ be a fundamental system of T-neighborhoods of e . For any subset M of X

$$(I_M)^- = \bigwedge_{\Gamma \in \mathfrak{R}} \Gamma I_M$$

Proof: From Proposition 3.8, $(I_M)^- \leq \Gamma I_M$ for every $\Gamma \in \mathfrak{R}$. Then

$$(I_M)^- \leq \bigwedge_{\Gamma \in \mathfrak{O}} \Gamma I_M.$$

Next we prove that

$$\bigwedge_{\Gamma \in \mathfrak{R}} \Gamma I_M = \bigwedge_{\Gamma \in \mathfrak{R}^{\sim}} \Gamma I_M.$$

Since $\mathfrak{R} \subset \mathfrak{R}^{\sim}$, then

$$\bigwedge_{\Gamma \in \mathfrak{R}} \Gamma I_M \geq \bigwedge_{\Gamma \in \mathfrak{R}^{\sim}} \Gamma I_M$$

Also, let $\Gamma \in \mathfrak{R}^{\sim}$, for all $\varepsilon > 0$ there exists $\Gamma_\varepsilon \in \mathfrak{R}$ such that $\varepsilon + \Gamma \geq \Gamma_\varepsilon$

$$\varepsilon + \Gamma I_M \geq (\varepsilon + \Gamma) I_M \geq \Gamma_\varepsilon I_M \geq \bigwedge_{\Gamma \in \mathfrak{R}} \Gamma I_M$$

Since this holds for all $\varepsilon > 0$, then

$$\Gamma I_M \geq \bigwedge_{\Gamma \in \mathfrak{R}} \Gamma I_M.$$

This inequality holds for all $\Gamma \in \mathfrak{R}^{\sim}$.

Consequently,

$$\bigwedge_{\Gamma \in \mathfrak{R}^{\sim}} \Gamma I_M \geq \bigwedge_{\Gamma \in \mathfrak{R}} \Gamma I_M$$

Hence, equality holds.

It is clear that if $\mathcal{O}$ is the set of symmetric elements in $\mathfrak{R}^{\sim}$ then.

$$\bigwedge_{\Gamma \in \mathfrak{R}} \Gamma I_M = \bigwedge_{\Gamma \in \mathfrak{R}^{\sim}} \Gamma I_M \leq \bigwedge_{\Delta \in \mathcal{O}} \Delta I_M$$

Conversely, let $\mathcal{O}$ is the set of symmetric elements in $\mathfrak{R}^{\sim}$. Then $\mathcal{O}$ is a fundamental system at e :

$$\begin{aligned} (\bigwedge_{\Delta \in \mathcal{O}} \Delta I_M)(x) &= \inf_{\Delta \in \mathcal{O}} (\Delta I_M)(x) \\ &= \inf_{\Delta \in \mathcal{O}} \sup_{y \in X} I_M(y) T (\Delta I_x)(y) \quad \text{by Proposition 3.7} \\ &= (I_M)^-(x) \end{aligned}$$

because the set $\{\Delta I_x : \Delta \in \mathcal{O}\}$ is a T-neighborhood basis at x .

Theorem 3.3. A T-G-action topology on X is a T-regular T-neighborhood topology.

Proof: Let $M \subset X$ and $x \in X$. We establish condition $(N^4\text{-T-regularity})$ of Theorem 3.2 in [6], which is equivalent to the T-regularity of X . For all

$M \subset X, x \in X$ such that

$$\begin{aligned} \text{Inf hgt } (\rho \text{ T } v: \rho \in \mathcal{C}(M), v \in OI_x) &\leq \inf_{\Delta \in \mathcal{O}} \text{hgt } (\Delta I_M T \Delta I_x) \\ &\leq \inf_{\Delta \in \mathcal{O}} \sup \{((\Delta I_M) \wedge \Delta I_x)(y): y \in X\} \\ &= \inf_{\Delta \in \mathcal{O}} \sup \{((\Delta I_M)(y) \wedge (\Delta I_x)(y)): y \in X\} \\ &= \inf_{\Delta \in \mathcal{O}} \sup \{ \sup \{ \Delta(h): h \in G, hy \in M \} \\ &\quad \wedge \sup \{ \Delta(k): k \in G, y = kx \}: y \in X \} \\ &= \inf_{\Delta \in \mathcal{O}} \sup \{ \sup \{ \Delta(h): h \in G, hy \in M \} \wedge \\ &\quad \sup \{ \Delta(k): k \in G, y = kx \}: y \in \text{orbit } x \} \end{aligned}$$

So. (call $y \in kx$)

$$\begin{aligned} \text{Inf hgt } (\rho \text{ T } v: \rho \in \mathcal{C}(M), v \in OI_x) &\leq \inf_{\Delta \in \mathcal{O}} \sup \{ \sup \{ \Delta(h): h k x \in M \} \wedge \Delta(k): k \in G \} \\ &= \inf_{\Delta \in \mathcal{O}} \sup \{ \Delta(h) \wedge \Delta(k): h, k \in G \text{ and } h k x \in M \} \\ &= \inf_{\Delta \in \mathcal{O}} \sup \{ (\Delta \Delta)(g): g \in G \text{ and } g x \in M \} \\ &= \inf_{\Delta \in \mathcal{O}} \sup \{ (\Delta \Delta)(I_M)(x) \} \end{aligned}$$

But by Theorem 2.2 in [7], for every $\Delta \in \mathcal{O}$, $\varepsilon \geq 0$ there exists $\Delta_1 \in \mathcal{O}$ such that $\Delta_1 \Delta_1 \leq \Delta + \varepsilon$. Hence,

$$\begin{aligned} \text{Inf hgt } (\rho \text{ T } v: \rho \in \mathcal{C}(M), v \in OI_x) &\leq \inf_{\Delta \in \mathcal{O}, \varepsilon > 0} ((\Delta + \varepsilon) I_M)(x) \\ &= \inf_{\Delta \in \mathcal{O}} (\Gamma I_M)(x) \\ &= \bigwedge_{\Gamma \in \mathcal{O}} (\Gamma I_M)(x) = (I_M)^-(x) \quad \text{by Proposition 3.9.} \end{aligned}$$

The opposite inequality is always valid.

Theorem 3.4. A T-G-action-topology τ_X^{T-G} coincides with the final T-neighborhood topology τ_f on X defined by the set of functions

$$\{\hat{x}: G \rightarrow X: x \in X, \hat{x}(g) = gx\}$$

Proof: For any $x \in X$, the function

$\hat{x}: G \rightarrow (X, \tau_X^{T-G})$ is continuous, because for all $g \in G$ and for each neighborhood $\Gamma(I_{gx})$ in the fundamental system $\mathfrak{R}_{I_{gx}}$ of $\hat{x}(g) = gx$, where $\Gamma \in \mathfrak{R}$, we have

$$\begin{aligned} \hat{x}(\Gamma \cdot I_g)(y) &= \sup \{ (\Gamma \cdot I_g)(h): h \in G, \hat{x}(h) = y \} \\ &= \sup \{ (\Gamma \cdot I_g)(h): h \in G, hx = y \} \\ &= (\Gamma \cdot I_g) I_x(y) \end{aligned}$$

then $\hat{x}(\Gamma \cdot I_g) = (\Gamma \cdot I_g) I_x = \Gamma I_{gx}$ and ΓI_{gx} is a T-neighborhood of g by Theorem 2.3 in [7]. Therefore

$\tau_X^{T-G} \subset \tau_f$ since τ_f is the finest T-neighborhood

topology making all $\hat{x}$ continuous.

Next, let $x \in X$, μ a T-neighborhood of x in τ_f . Then $\hat{x}^{-1}(\mu)$ a T-neighborhood of e in G ; i.e. $(\hat{x}^{-1}(\mu))I_x$ is a T-neighborhood of x in τ_X^{T-G} .

$$\text{But } (\hat{x}^{-1}(\mu))I_x = \hat{x}(\hat{x}^{-1}(\mu)) = \mu \wedge 1_{\text{range}\hat{x}} \leq \mu.$$

This proves that μ is a T-neighborhood of x in τ_X^{T-G} .

Then $\tau_f \subset \tau_X^{T-G}$. Hence, equality holds.

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