

# Global Existence of Periodic Solutions in a Delayed Tri-neuron Network

Kejun Zhuang, Zhaohui Wen

**Abstract**—In this paper, a tri-neuron network model with time delay is investigated. By using the Bendixson's criterion for high-dimensional ordinary differential equations and global Hopf bifurcation theory for functional differential equations, sufficient conditions for existence of periodic solutions when the time delay is sufficiently large are established.

**Keywords**—Delay, global Hopf bifurcation, neural network, periodic solutions.

## I. INTRODUCTION

IN the past decade, dynamical behaviors of delayed neural networks have been extensively investigated. For instance, Hopf bifurcation in delayed differential equations modeling three-neuron was explicitly studied in [1–2]. The global asymptotic stability of Hopfield neural networks with delays was considered by utilizing Lyapunov functionals [3–4]. Existence and exponential stability of periodic solutions and almost periodic solutions for cellular neural networks were established by using the fixed point theorem and differential inequality techniques [5–6]. Moreover, the Hopf bifurcation in discrete cases of Hopfield-type  $n$ -dimension neural network model was proved in [7]. However, there are still a lot more work to do on the bifurcation of these models, especially the global continuation of local Hopf bifurcation.

In this paper, we consider the following tri-neuron network with a delay:

$$\begin{cases} \dot{u}_1(t) = -a_1 u_1(t) + w_{11} f(u_1(t)) + w_{12} f(u_2(t)) \\ \quad + w_{13} f(u_3(t - \tau)), \\ \dot{u}_2(t) = -a_2 u_2(t) + w_{21} f(u_1(t)) + w_{22} f(u_2(t)) \\ \quad + w_{23} f(u_3(t)), \\ \dot{u}_3(t) = -a_3 u_3(t) + w_{32} f(u_2(t)) + w_{33} f(u_3(t)), \end{cases} \quad (1)$$

where  $u_i (i = 1, 2, 3)$  is the activation of neuron  $i$ .  $a_i > 0 (i = 1, 2, 3)$  is the decay rate of neurons,  $w_{ij}$  is the weight of synaptic connections from neuron  $j$  to neuron  $i$ ,  $\tau > 0$  is the synaptic transmission delay and  $f(\cdot)$  is the activation function. Liu et al. [1] discussed the necessary and sufficient conditions for Hopf bifurcation from the nonzero equilibrium of (1) by taking the time delay as a bifurcation parameter.

The purpose is to establish the global existence of Hopf bifurcating periodic solutions for (1) based on the Bendixson's criterion for high-dimensional ordinary differential equations [8] and the global bifurcation theory due to Wu [9].

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The rest of this paper is organized as follows: in next section, the existence of local Hopf bifurcation is stated. In Section 3, global continuation of local existence of periodic solutions is obtained.

## II. PRELIMINARIES

For convenience, we first elaborate the stability and bifurcation structure for system (1), which can be found in [1].

Let  $E^* = (u_1^*, u_2^*, u_3^*)$  denote the nonzero equilibrium, the corresponding characteristic equation at  $E^*$  is in the form

$$\lambda^3 + b_1 \lambda^2 + b_2 \lambda + b_3 = A e^{-\lambda \tau}, \quad (2)$$

where  $b_1 = a_1 + a_2 + a_3 - w_{11} f'(u_1^*) - w_{22} f'(u_2^*) - w_{33} f'(u_3^*)$ ,  $b_2 = [a_2 - w_{33} f'(u_3^*)][a_1 + a_2 - w_{11} f'(u_1^*) - w_{22} f'(u_2^*)] - w_{23} w_{32} f'(u_2^*) f'(u_3^*) - w_{12} w_{21} f'(u_1^*) f'(u_2^*)$ ,  $b_3 = [a_1 - w_{11} f'(u_1^*)][a_2 - w_{22} f'(u_2^*)][a_3 - w_{33} f'(u_3^*)] - w_{23} w_{32} f'(u_2^*) f'(u_3^*) [a_1 - w_{11} f'(u_1^*)] - w_{12} w_{21} f'(u_1^*) f'(u_2^*) [a_3 - w_{33} f'(u_3^*)]$ ,  $A = -w_{13} w_{21} w_{32} f'(u_1^*) f'(u_2^*) f'(u_3^*)$ .

If  $\lambda = \pm i\omega (\omega > 0)$  are characteristic roots, then we can rewrite (2) in terms of

$$\omega^6 + (b_1^2 - 2b_2)\omega^4 + (b_2^2 - 2b_1 b_3)\omega^2 + b_3^2 - A^2 = 0.$$

**Lemma 2.1.** Let  $d_1 = b_1^2 - 2b_2$ ,  $d_2 = b_2^2 - 2b_1 b_3$ ,  $d_3 = b_3^2 - A^2$ ,  $\Delta = \frac{4}{27} d_3^3 - \frac{1}{27} d_1^2 d_2^2 + \frac{4}{27} d_1^3 d_3 - \frac{2}{3} d_1 d_2 d_3 + d_2^3$ . Suppose  $d_3 > 0$  and make the following assumptions:

(H1)  $d_1 < 0$ ,  $d_2 \geq 0$ ,  $d_1^2 > 3d_2$ ,  $\Delta < 0$ ;

(H2)  $d_2 < 0$ ,  $\Delta < 0$ ;

(H3)  $3d_3 > d_1^2$ ;

(H4)  $3d_2 = d_1^2$ ;

(H5)  $d_1^2 > 3d_2$ ,  $\Delta > 0$ ;

(H6)  $d_1^2 > 3d_2$ ,  $\Delta \leq 0$ ,  $d_1 > 0$ ,  $d_2 > 0$ .

If either (H1) or (H2) holds, then (2) has a pair of purely imaginary roots  $\pm i\omega_0$  when  $\tau = \tau_j = \frac{1}{\omega_0} \left[ \arccos \frac{b_3 - b_1 \omega_0^2}{A} + 2j\pi \right]$ ,  $j = 0, 1, 2, \dots$ . If one of (H3), (H4), (H5) and (H6) is satisfied, then equation (2) has no purely imaginary root.

**Lemma 2.2.**  $\left. \frac{d \operatorname{Re} \lambda(\tau)}{d\tau} \right|_{\tau=\tau_j} > 0$ .

**Theorem 2.3.** If one of (H3), (H4), (H5) and (H6) is satisfied, then equilibrium  $E^*$  of (1) is stable for any  $\tau > 0$ . If either (H1) or (H2) holds, then  $E^*$  is locally asymptotically stable when  $\tau \in [0, \tau_0)$  and unstable when  $\tau > \tau_0$ , Hopf bifurcation occurs as  $\tau$  passes through  $\tau_0$ .

III. GLOBAL EXISTENCE OF PERIODIC SOLUTIONS

In this section, we shall show the global continuation of positive periodic solutions bifurcating from the equilibrium  $E_*$ . Throughout this section, we closely follow the notation in [9] and make the following definitions

$$X = C([-τ, 0], \mathbf{R}^3),$$

$\Sigma = Cl\{(x, \tau, p) : (x, \tau, p) \in X \times \mathbf{R}_+ \times \mathbf{R}_+, x \text{ is a } p\text{-periodic solutions of (1)}\}$ ,

$$\Delta = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 - Ae^{-\lambda\tau},$$

and let  $C(x^*, \tau_j, 2\pi/\omega_0)$  denote the connected component of  $(x^*, \tau_j, 2\pi/\omega_0)$  in  $\Sigma$ , where  $\omega_0$  and  $\tau_j$  are defined in Lemma 2.1.

**Lemma 3.1.** If  $f(\cdot)$  is bounded, then all periodic solutions of (1) are uniformly bounded.

**Proof.** Let  $a = \min\{a_1, a_2, a_3\}$ ,  $f(\cdot) < L$ ,  $M \geq \max\{1, 3L(w_1+w_2+w_3)/a\}$ ,  $w_1 = \max\{|w_{11}|, |w_{12}|, |w_{13}|\}$ ,  $w_2 = \max\{|w_{21}|, |w_{22}|, |w_{23}|\}$ ,  $w_3 = \max\{|w_{32}|, |w_{33}|\}$  and  $r(t) = \sqrt{u_1^2(t) + u_2^2(t) + u_3^2(t)}$ . Differentiating  $r(t)$  along a solution of (1), we have

$$\begin{aligned} \dot{r}(t) &= \frac{1}{r(t)}[-a_1u_1^2(t) + w_{11}u_1(t)f(u_1(t)) + \\ &w_{12}u_1(t)f(u_2(t)) + w_{13}u_1(t)f(u_3(t-\tau)) - a_2u_2^2(t) \\ &+ w_{21}u_2(t)f(u_1(t)) + w_{22}u_2(t)f(u_2(t)) + \\ &w_{23}u_2(t)f(u_3(t)) - a_3u_3^2(t) + w_{32}u_3(t)f(u_2(t)) \\ &+ w_{33}u_3(t)f(u_3(t))] \\ &\leq \frac{1}{r(t)}[-a(u_1^2(t) + u_2^2(t) + u_3^2(t)) + 3L(w_1|u_1(t)| \\ &+ w_2|u_2(t)| + w_3|u_3(t)|)]. \end{aligned}$$

If there exists  $t_0 > 0$  such that  $r(t_0) = A > M$  and from the inequality  $w_1|u_1| + w_2|u_2| + w_3|u_3| \leq (w_1 + w_2 + w_3)\sqrt{u_1^2 + u_2^2 + u_3^2}$ , then we get

$$\begin{aligned} \dot{r}(t_0) &\leq \frac{1}{A}[-aA^2 + 3AL(w_1 + w_2 + w_3)] \\ &= -aA + 3L(w_1 + w_2 + w_3) \\ &< 0. \end{aligned}$$

It follows that if  $u(t) = (u_1(t), u_2(t), u_3(t))$  is a periodic solution of (1), then  $r(t) < M$  for any  $t > 0$ . Hence, the periodic solutions of (1) are uniformly bounded.

For simplicity, we make the assumption as follows:

(H7) There exist positive constants  $\alpha$  and  $\beta$ , such that  $\sup_{x \in \mathbf{R}} \{- (a_1 + a_2) + w_{11}|f'(u_1)| + w_{22}|f'(u_2)| + \frac{\alpha}{\beta}w_{23}|f'(u_3)| + \alpha w_{13}|f'(u_3)|, -a_1 + w_{11}|f'(u_1)| + w_{22}|f'(u_2)| + \frac{\beta}{\alpha}w_{32}|f'(u_2)| + w_{22}|f'(u_2)|, -(a_2 + a_3) + w_{22}|f'(u_2)| + w_{33}|f'(u_3)| + \frac{1}{\beta}w_{21}|f'(u_1)|\} < 0$ .

**Lemma 3.2.** If (H7) is satisfied, then (1) has no nonconstant  $\tau$ -periodic solution.

**Proof.** For contradiction, we suppose that system (1) has nonconstant  $\tau$ -periodic solutions, then the following ordinary differential system has nonconstant periodic solutions:

$$\begin{cases} \dot{u}_1(t) = -a_1u_1(t) + w_{11}f(u_1(t)) + w_{12}f(u_2(t)) \\ \quad + w_{13}f(u_3(t)), \\ \dot{u}_2(t) = -a_2u_2(t) + w_{21}f(u_1(t)) + w_{22}f(u_2(t)) \\ \quad + w_{23}f(u_3(t)), \\ \dot{u}_3(t) = -a_3u_3(t) + w_{32}f(u_2(t)) + w_{33}f(u_3(t)). \end{cases} \quad (3)$$

Denote  $u = (u_1, u_2, u_3)^T$ ,  $F(u_1, u_2, u_3) = (-a_1u_1 + w_{11}f(u_1) + w_{12}f(u_2) + w_{13}f(u_3), -a_2u_2 + w_{21}f(u_1) + w_{22}f(u_2) + w_{23}f(u_3), -a_3u_3 + w_{32}f(u_2) + w_{33}f(u_3))^T$ . We have the second additive compound matrix [8] as follows

$$\frac{\partial F^{[2]}}{\partial u} = \begin{pmatrix} a_{11} & w_{23}f'(u_3) & -w_{13}f'(u_3) \\ w_{32}f'(u_2) & a_{22} & -a_2 + w_{22}f'(u_2) \\ 0 & w_{21}f'(u_1) & a_{33} \end{pmatrix}.$$

The second compound system takes the form

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \frac{\partial F^{[2]}}{\partial u} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

Let  $W(z) = \max\{\alpha|z_1|, \beta|z_2|, |z_3|\}$ ,  $\alpha > 0, \beta > 0$ . Direct calculation leads to the following inequalities

$$\begin{aligned} \frac{d^+}{dt} \alpha|z_1(t)| &\leq -(a_1 + a_2)\alpha|z_1| + (w_{11}|f'(u_1)| \\ &+ w_{22}|f'(u_2)|)\alpha|z_1| + \frac{\alpha}{\beta}w_{23}|f'(u_3)|\beta|z_2| \\ &+ \alpha w_{13}|f'(u_3)||z_3|. \\ \frac{d^+}{dt} \beta|z_2(t)| &\leq -(a_1 + a_2)\beta|z_2| + (w_{11}|f'(u_1)| \\ &+ w_{22}|f'(u_2)|)\beta|z_2| + \frac{\beta}{\alpha}w_{32}|f'(u_2)|\alpha|z_1| \\ &+ (a_2 + w_{22}|f'(u_2)|)|z_3|. \\ \frac{d^+}{dt} |z_3(t)| &\leq -(a_2 + a_3)|z_3| + (w_{22}|f'(u_2)| + w_{33}|f'(u_3)|)|z_3| \\ &+ \frac{1}{\beta}w_{21}|f'(u_1)|\beta|z_2|. \end{aligned}$$

Therefore,

$$\frac{d^+}{dt} W(z(t)) \leq \mu(t)W(z(t)),$$

where  $\mu(t) = \max\{- (a_1 + a_2) + w_{11}|f'(u_1)| + w_{22}|f'(u_2)| + \frac{\alpha}{\beta}w_{23}|f'(u_3)| + \alpha w_{13}|f'(u_3)|, -a_1 + w_{11}|f'(u_1)| + w_{22}|f'(u_2)| + \frac{\beta}{\alpha}w_{32}|f'(u_2)| + w_{22}|f'(u_2)|, -(a_2 + a_3) + w_{22}|f'(u_2)| + w_{33}|f'(u_3)| + \frac{1}{\beta}w_{21}|f'(u_1)|\}$ . By (H7), there exists  $\delta > 0$  such that  $\mu(t) \leq -\delta < 0$ . Hence  $W(z(t)) \leq W(z(s))e^{-\delta(t-s)}$  when  $t \geq s > 0$ . This establishes the equiuniform asymptotic stability of the second compound system. This completes the proof.

**Theorem 3.3.** If  $f(\cdot)$  is bounded and (H7) holds, either (H1) or (H2) holds, then periodic solutions bifurcating from positive equilibrium of (1) still exist for  $\tau > \tau_j, j = 0, 1, 2, \dots$

**Proof.** It is only need to prove that the projection of  $C(x_*, \tau_j, 2\pi/\omega_0)$  onto  $\tau$ -space includes  $[\tau_j, \infty)$  for each  $j \geq 0$ .

From Lemma 2.1 and 2.2, there exist  $\varepsilon > 0, \delta > 0$  and a smooth curve  $\lambda : (\tau_j - \delta, \tau_j + \delta) \rightarrow \mathbf{C}$ , such that

$$\Delta(\lambda(\tau)) = 0, \quad |\lambda(\tau) - i\omega_0| < \varepsilon,$$

for all  $\tau \in [\tau_j - \delta, \tau_j + \delta]$  and

$$\lambda(\tau_j) = i\omega_0, \quad \left. \frac{d\text{Re}(\lambda(\tau))}{d\tau} \right|_{\tau=\tau_j} > 0.$$

Define  $\Omega_\varepsilon = \{(u, p) : 0 < u < \varepsilon, |p - 2\pi/\omega_0| < \varepsilon\}$ . It is not difficult to show that if  $|\tau - \tau_j| \leq \delta$  and  $(u, p) \in \Omega_\varepsilon$ , then  $\Delta(x_*, \tau, p)(u + 2im\pi/p) = 0$  if and only if  $u = 0, \tau = \tau_j, p = 2\pi/\omega_0$ . This verifies the assumptions (A1)–(A4) in [9] for  $m = 1$ .

Moreover, putting

$$H^\pm \left( x_*, \tau_j, \frac{2\pi}{\omega_0} \right) (u, p) = \Delta(x_*, \tau_j \pm \delta, p) \left( u + i \frac{2\pi}{p} \right),$$

then we can compute the crossing number of the isolated center  $(x_*, \tau_j, 2\pi/\omega_0)$  as follows

$$\begin{aligned} \gamma \left( x_*, \tau_j, \frac{2\pi}{\omega_0} \right) &= \deg_B \left( H^- \left( x_*, \tau_j, \frac{2\pi}{\omega_0} \right), \Omega_\varepsilon \right) \\ &\quad - \deg_B \left( H^+ \left( x_*, \tau_j, \frac{2\pi}{\omega_0} \right), \Omega_\varepsilon \right) \\ &= -1, \end{aligned}$$

where  $\deg_B$  denotes the Brouwer degree. Then we have

$$\sum_{(\hat{x}, \tau, p) \in C(x_*, \tau_j, 2\pi/\omega_0)} \gamma(\hat{x}, \tau, p) < 0.$$

Therefore, from Theorem 3.3 in [9], the connected component  $C(x_*, \tau_j, 2\pi/\omega_0)$  in  $\Sigma$  is unbounded.

Lemma 3.1 implies that the projection of  $C(x_*, \tau_j, 2\pi/\omega_0)$  onto  $x$ -space is bounded. From the definition of  $\tau_j$ , we know that  $0 < 2\pi/\omega_0 < \tau_j$  when  $j > 0$ . Then the projection onto  $p$ -space is also bounded.

Besides, the projection of  $C(x_*, \tau_j, 2\pi/\omega_0)$  onto  $\tau$ -space is bounded below due to Lemma 3.2. This means that the projection of  $C(x_*, \tau_j, 2\pi/\omega_0)$  onto  $\tau$ -space must be unbounded and includes  $[\tau_j, \infty)$ . As a result, bifurcating periodic solutions of (1) still exist when  $\tau$  is far away from the first critical value  $\tau_0$ .

#### IV. CONCLUSION

This paper proves the global existence of Hopf bifurcation for a tri-neuron network with time delay. The main theorem shows that local bifurcation may mean the global bifurcation under certain condition. Moreover, the activation function in neural network is usually hyperbolic tangent and the conditions of Theorem 3.3 can be satisfied. Thus, the results are new and complement previously known results.

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