Fast Segmentation for the Piecewise Smooth Mumford-Shah Functional

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Abstract—This paper is concerned with an improved algorithm based on the piecewise-smooth Mumford and Shah (MS) functional for an efficient and reliable segmentation. In order to speed up convergence, an additional force, at each time step, is introduced further to drive the evolution of the curves instead of only driven by the extensions of the complementary functions u^+ and u^-. In our scheme, furthermore, the piecewise-constant MS functional is integrated to generate the extra force based on a temporary image that is dynamically created by computing the union of u^+ and u^- during segmenting. Therefore, some drawbacks of the original algorithm, such as smaller objects generated by noise and local minimal problem also are eliminated or improved. The resulting algorithm has been implemented in Matlab and Visual C++, and demonstrated efficiently by several cases.

Keywords—Active contours, energy minimization, image segmentation, level sets.

I. INTRODUCTION

VARIATIONAL-based image segmentation method has become one of the most important steps in analyzing image data. Variational techniques can be divided into three groups: diffusion-based techniques, curve evolution techniques and techniques based on region models. Diffusion-based techniques are based on diffusing information from a pixel to its neighbors in the image. This usually results in a smoothing image. Curve evolution techniques attempt to evolve a closed contour over the image domain. Active contours or "snakes" can be used to segment object automatically. This framework has been used successfully by Kass et al [2] to extract boundaries and edges. One potential problem with this approach is that the topology of the region to be segmented must be known in advance. An algorithm to overcome these difficulties was first introduced by Osher and Sethian [4]. They model the propagating curve as a specific level set of a higher dimensional surface. However, this method has proved erroneous when in particular the image is noisy or its edges are not clear. Variational models for image segmentation have had great success. One of the most successful and pioneering models that adopt this approach is the model of Mumford and

Shah (MS) [1]. Based on the MS segmentation technique, Chan and Vese [5, 6] proposed an active contour model without edge, and the model can detect edges both with and without gradients from objects that are smooth or even have discontinuous boundaries. However, the MS model in piecewise-constant case cannot detect objects successfully from noisy images. To overcome the drawback, Chan and Vese [6] showed how the piecewise-smooth MS segmentation problem can be solved using the level set method, and they have given the piecewise-smooth optimal approximations of a given image. Although piecewise-smooth MS model works better, however, It requires the initial curve to be close to the boundaries, otherwise the convergence of the curve to object boundary will be too slow, and for highly noisy images, it almost collapses. In this paper, we propose a efficient partial difference equation (PDE)-based algorithm for solving the low convergence problem of the piecewise-smooth MS segmentation functional. The extensions of complementary functions u^+ and u^- don't have to be computed while the segmenting. Instead, the level set function is first updated in advance based on an assembled image. The assembled image can be regarded as an intermediate version of the original image, so the evolution of curves can be performed on it to adjust the pose and provide an additional drive force to speed up convergence. The piecewise-constant MS algorithm is applied as adjust function to provide an additional drive force. So, the resulting algorithm has some advantages of piecewise-constant MS model, such as with faster speed of evolution of curves and better in edge preserving properties, the evolution of curves is independent of the choice of the initial curve.

The rest of this paper is organized as follows. The Mumford and Shah model is introduced in section 2, and section 3 describes the proposed algorithm. Some results of numeric experiments are given in Section 4, which is followed by concluding remarks in Section 5.

II. MUFORD AND SHAH MODEL

The Mumford and Shah model is a variational problem for approximating a given image by a piecewise smooth image of minimal complexity. Let $\Omega \in R^N$ be a bound domain with Lipschitz boundary, modeling the image domain. Let $u_0 : \Omega \to R$ represent a grayscale image. To find the segmentation Γ of u_0, Mumford-Shah piecewise smooth

segmentation [1] is defined to carry out the following minimization:

$$\inf_{u,\Gamma} E_{MS}(u,\Gamma \mid u_0) = \int_{\Omega}(u-u_0)^2 dx$$
$$+ \mu \int_{\Omega \setminus \Gamma}|\nabla u|^2 dx + v \mid \Gamma \mid \qquad (1)$$

where μ and v are positive parameters, u is the image intensity. It allows the segmented "objects" to have smoothly varying intensities. Chan-Vese [6] showed how the piecewise-smooth MS segmentation problem can be solved using the level set method. In their model, two functions u^+ and u^- are introduced, such that:

$$u(x) = u^+(x)H(\phi(x)) + u^-(x)(1 - H(\phi(x))) \qquad (2)$$

where $H(z)$ is Heaviside function, In Chan-Vese algorithm [5], the authors regularized it as

$$H(z) = \frac{1}{2}\left(1 + \frac{2}{\pi}\arctan(\frac{z}{\varepsilon})\right) \qquad (3)$$

Those two functions u^+ and u^- are assumed that are C^1 functions on $\phi \geq 0$ and $\phi < 0$ respectively, and with continuous derivatives up to all boundary points, i.e. up to the boundary $\{\phi = 0\}$. Substituting this expression (2) into (1), one can obtain:

$$\inf_{u^+,u^-,\Phi|u_0} E(u^+,u^-,\phi \mid u_0) = \int_{\Omega}|u^+-u_0|^2 H(\phi)dx$$
$$+ \int_{\Omega}|u^--u_0|^2(1-H(\phi))dx + \mu\int_{\Omega}|\nabla u^+|^2 H(\phi)dx \quad (4)$$
$$+ \mu\int_{\Omega}|\nabla u^-|^2(1-H(\phi))dx + v\int_{\Omega}|\nabla H(\phi)|$$

Then with ϕ fixed, the equation (4) leads to the two Euler-Lagrange equations for u^+ and u^- can be written as

$$\begin{cases} u^+ - u_0 = \mu\Delta u^+ & \{(x):\phi(x,t)>0\} \\ \dfrac{\partial u^+}{\partial \vec{n}} = 0 & \{(x):\phi(x,t)=0\}\cup\partial\Omega \\ u^- - u_0 = \mu\Delta u^- & \{(x):\phi(x,t)<0\} \\ \dfrac{\partial u^-}{\partial \vec{n}} = 0 & \{(x):\phi(x,t)=0\}\cup\partial\Omega \end{cases} \qquad (5)$$

Notice that u^+ and u^- act as denosing operators on the homogeneous regions only. No smoothing is done across the boundary $\{\phi = 0\}$, which is very important in image analysis.

Now, keeping u^+ and u^- fixed, and minimizing $E_{MS}(u^+,u^-,\phi \mid u_0)$ with respect to the function ϕ, one can obtain the motion of the zero level set as following:

$$\frac{\partial \phi}{\partial t} = \delta(\phi)(v\nabla(\frac{\nabla \phi}{|\nabla \phi|}) - |u^+-u_0|^2 - \mu|\nabla u^+|^2$$
$$+ |u^--u_0|^2 + \mu|\nabla u^-|^2) \qquad (6)$$

where the delta function is defined as the derivative of the Heaviside function:

$$\delta(z) = \frac{1}{\pi}\left(\frac{\varepsilon}{\varepsilon^2 + z^2}\right) \qquad (7)$$

The above equation (6) with some initial guess ϕ (t=0, x) is actually computed at least near a narrow band of the zero level set. As a result, computationally, one have to continuously extended both u^+ and u^- from their original domain $\{\pm\phi > 0\}$ to a suitable neighborhood of the zero level set $\{\phi = 0\}$. The extension of u^+ and u^- is important to curve convolution. Furthermore, u^+ and u^- can be easily obtained by solving Euler-Lagrange equations (5). However, the extensions of both u^+ and u^- are very difficult, it requires solving the following degenerate elliptic linear equations:

$$\begin{cases} u_t^+ = \nabla^2 u^+(\vec{N},\vec{N}) & \{\phi < 0\} \\ \dfrac{\partial u^+}{\partial \vec{n}} = 0 \end{cases} \qquad (8)$$

and

$$\begin{cases} u_t^- = \nabla^2 u^-(\vec{N},\vec{N}) & \{\phi > 0\} \\ \dfrac{\partial u^-}{\partial \vec{n}} = 0 \end{cases} \qquad (9)$$

Chan and Vese [5] have pointed out three possible ways to solve the problem, but all of them are difficult to carry out in practice. So in this paper, a new strategy is proposed to attack the problem. It will be described in following sections.

III. DESCRIPTION OF THE PROPOSED ALGORITHM

In this context, the aim is to speed up the convergence and edge preserving. It has been known that, in the piecewise-smooth MS problem, the extensions of two functions u^+ and u^- can be crucial factors for evolution of curves. In [6][8], they are given as following:

$$u_{i,j}^{+,n+1} = (u_{i-1,j}^{+,n} + u_{i+1,j}^{+,n} + u_{i-1,j-1}^{+,n} + u_{i+1,j+1}^{+,n})/4 \qquad (10)$$

The corresponding u^- can be obtained by replacing u^+ with u^-. Since the evolution of the curves coupled with diffusion in the piecewise-smooth MS method, the evolution of the curves gets very slow as the diffusion progresses. The reason is twofold. First of all, it is due to the fact that with the diffusion and the noise removed from an image, the regions of the image become more and more homogeneous, and their boundaries are burred. Secondly, it is the reason that the evolution speed strongly depends on the solutions of the two complementary functions u^+ and u^-, but they couldn't be solved accurately and efficiently. Although the authors, in [6], suggested three different methods to solve u^+ and u^-, none of them is accurate and efficient. Up to now, researchers still haven't found a better way for solutions of u^+ and u^-.

To attack these problems, one may consider strategies: one is re-initialization of the level set function in each time step. The other is to seek some new methods for a better solutions of u^+ and u^-.

A. Reinitialization of the Level Set Function for Speeding up Convergence

In the piecewise-smooth MS algorithm, the evolution of curves coupled with diffusion, so the image might have become very homogeneous in certain iterations. Furthermore, the evolution speed of curves is reduced greatly. In order to speed up convergence, we can further modify or adjust the location of the curves by an additional force during evolution of curves. The aim of modification is to update the level set function ϕ to a better location. For the purpose, a lot of methods for active contour, in theory, could be applied. Note that the advantage of the piecewise-constant MS functional is to work better for homogeneous regions and, in theory, robust. Therefore, it is very appropriate to be used to generate an extra force and update ϕ efficiently. The input parameters of the algorithm include both u^+ and u^-, the current value of ϕ and a temporary image that will be introduced in following subsection. Then the new position of ϕ, modified by the extra force, is used as output of this algorithm. Finally, the updated version of ϕ, used as initial parameters, is applied to the piecewise-smooth MS algorithm again.

B. Alternative Solutions for Extension of u^+ and u^-

In this subsection, we introduce a new strategy for the solutions of u^+ and u^-. First of all, the major function of u^+ and u^- is de-noising noise. Because Eq. (8) and (9) are difficult to be solved by numerical method, extensions of u^+ and u^- usually are approximately computed by the equation (10). However, it results in some problems such as the lower convergence, local minimal problem, and so on. In our scheme, to solve the problem, a new strategy is proposed to drive directly the evolution of curves by an external force so as to avoid solving extensions of u^+ and u^-. To drive the evolution of curves to a new location by an additional force, all kinds of methods for active contour, in theory, could be applied. However, for simple, we wish the extended level set functions still were represented by distance function. By now, we still

haven't found an efficient method to transform a given complex active contours into distance function. Therefore this restriction against some active contour methods used in this case. In this context, the piecewise constant MS approach is chosen. Note that the piecewise-constant MS functional works better for homogeneous regions and, in theory, robust, hence it is the best appropriate candidate to be used for the purpose. As known in previous sections, to keep the evolution of curves, both u^+ and u^- have to be continuous extended from their original domain $\{\pm\phi > 0\}$ to a suitable neighborhood of the zero level set $\{\phi = 0\}$. Considering that u^+ and u^- act as a denoising operator on homogeneous regions outside or inside the boundaries $\{\phi = 0\}$, respectively, therefore a smoothing diffused image can be obtained by calculating the union of u^+ on $\{\phi > 0\}$ and u^- on $\{\phi < 0\}$. The intention is to directly evolve the level set function ϕ on the diffused image instead of the extensions of u^+ and u^-. As a result, some difficulties of the classical algorithms such as elimination of some smaller objects which is not corresponding to physical objects, solution of global minimal problem and so on can be overcome. The reason is that there are little noises in the working image, i.e. the temporary image since amount of noises have been removed.
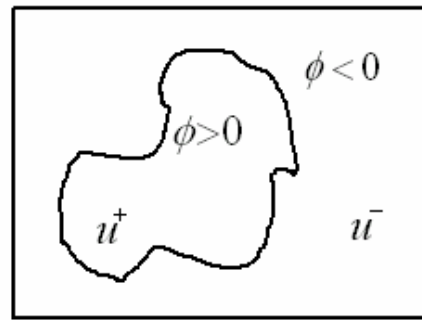

Fig. 1 The temporary image composed of the regions of $u^+ \cup u^-$

In addition, the temporary image, shown in Fig. 1, is defined based on a set operation of u^+ and u^-, represented as $u^+ \cup u^-$, with u^+ on $\phi > 0$ and u^- on $\phi < 0$, respectively. Such image, in proposed algorithm, is thus used to compute the new position of ϕ by the piecewise-constant MS algorithm. In other respects, if the diffused image gets too homogeneous or the external force is very strong, the updating procedure maybe makes ϕ away from the desired boundaries, even disappear from the image. Thus, for high-noise images, we have to limit the extra force at certain ranges. A simple solution for this problem can be obtained by limiting the times of iterations. Some better results, in our experiment, have been obtained with times of iteration less than 10.

C. Implementation

It was given that the principal steps of the numerical solution for the proposed algorithm, which can be outlined as following:

Initialization: define the initial level set function, with n=0. For each $n > 0$ until steady state:

1. Find $u_{i,j}^{n,+}$ on $\phi_{i,j}^n \geq 0$ and $u_{i,j}^{n,-}$ on $\phi_{i,j}^n \leq 0$ by (6)

2. Solve (10) for the extensions of $u_{i,j}^{n,+}$ on $\phi_{i,j}^n < 0$ and $u_{i,j}^{n,-}$ on $\phi_{i,j}^n > 0$, respectively.

3. Find the $\phi_{i,j}^{n+1}$, which is represented as $\hat{\phi}_{i,j}^{n+1}$, by solving the following partial differential equation:

$$\frac{\partial \phi}{\partial t} = \delta(\phi)(\nu \nabla (\frac{\nabla \phi}{|\nabla \phi|}) - |u^+ - u_0|^2 - \mu |\nabla u^+|^2$$
$$+ |u^- - u_0|^2 + \mu |\nabla u^-|^2$$

4. Let $\hat{u}_0 = u_{i,j}^{n,+} \{\phi_{i,j}^n \geq 0\} \bigcup u_{i,j}^{n,-} \{\phi_{i,j}^n < 0\}$ be the initial image, and $\hat{\phi}^0 = \hat{\phi}_{i,j}^{n+1}$ the initial level set function. Then $\phi_{i,j}^{n+1}$ can be obtained by [6]

$$\frac{\partial \phi}{\partial t} = \delta(\phi)\{\nu \cdot div(\frac{\nabla \phi}{|\nabla \phi|}) - (\hat{u}_0 - c_1)H(\phi)$$
$$+ (\hat{u}_0 - c_2)(1 - H(\phi))\}$$

where

$$c_1 = \frac{\int_\Omega \hat{u}_0 H(\phi)dxdy}{\int_\Omega H(\phi)dxdy}, \quad c_2 = \frac{\int_\Omega \hat{u}_0 (1 - H(\phi))dxdy}{\int_\Omega (1 - H(\phi))dxdy},$$

$\hat{u}_0$ denotes the temporary image, and $H(\phi)$ stands for the Heaviside function, with $\delta(\phi)$ as its derivative.

IV. NUMERIC EXPERIMENTS

In this section, some numerical results are given, all experiments are performed on Personal Computer and the algorithms are implemented with Visual C++ 6.0. For comparison we have used the following parameter values with the time step $\Delta t = 0.1$, space steps $h = \Delta x = \Delta y = 1$, $\mu = 1.0$, and $\nu = 0.0305 * 255^2$ in our experiment, which are the same with those of the classical algorithms [6].

Fig. 2 shows the segmentation of a blood vessel image of size 200×200 pixels, the top row shows the proposed algorithm, and the corresponding results segmented by the classical algorithm also are given at the bottom row. Note that the nonphysical components from the noises are decreased in the Proposed algorithm, thus in the classical method the considerable nonphysical components were introduced.
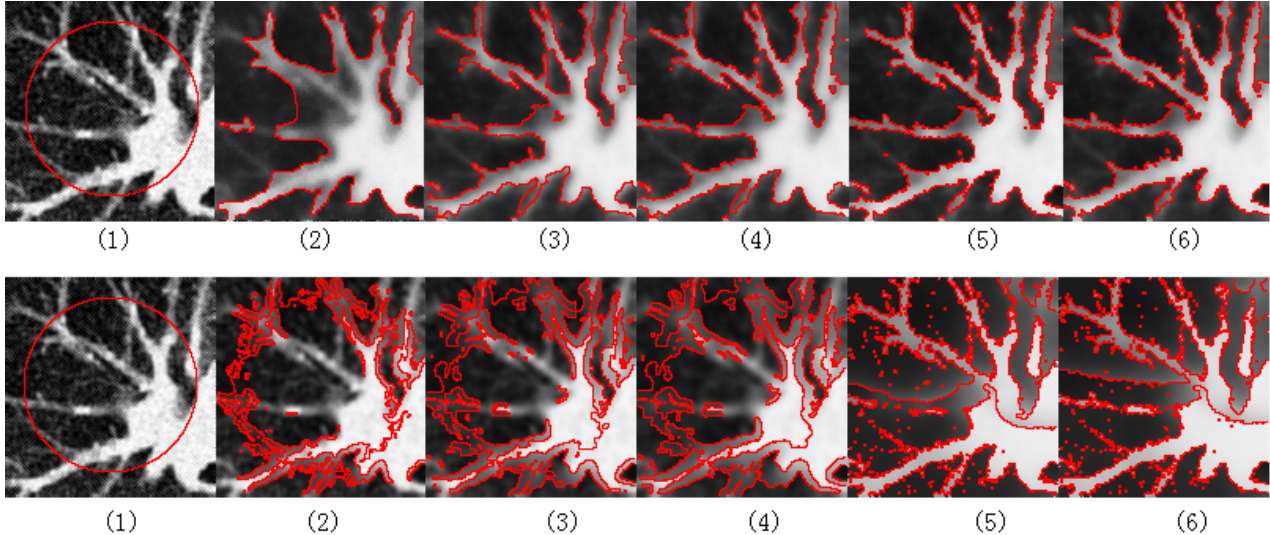

Fig. 2 Segmentation of a blood vessel image by the improved algorithm at top row, and by the standard algorithm at bottom
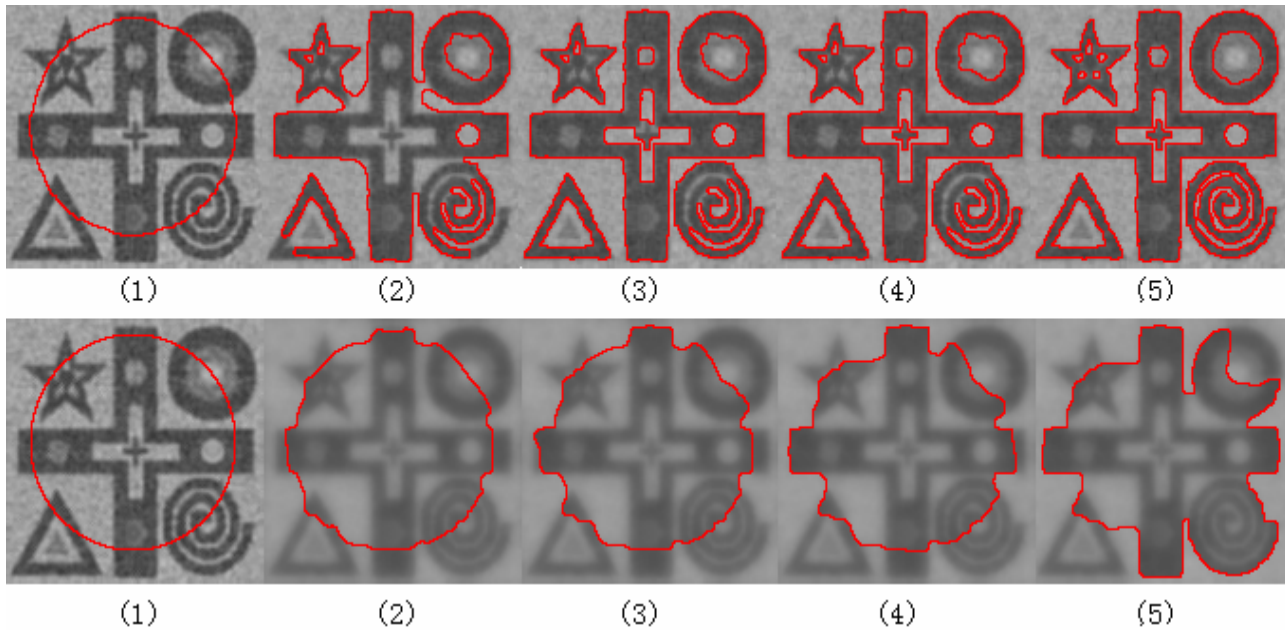
Fig. 3 Segmentation of an image corrupted with Gaussian noise by the improved algorithm at top row, and by the standard algorithm at bottom row

Fig. 3 shows how the new algorithm to work on an image corrupted with Gaussian noise. The top row shows the proposed algorithm, and the corresponding results obtained by the classical algorithm are given at the bottom row. Note that the classical algorithm did not convergence to segment it. Fig. 4 shows results of an experiment with a noisy image of size 128 ×128 pixels. The image is close to being piecewise constant; this makes it relatively easy to be segmented using the proposed method. Thus the classical method almost collapses on it. Fig. 5 demonstrates an advantage of the proposed approach in speeding up convergence. Only 162 iterations were necessary to segment the artificial image with furry edges, (Fig. 5a) by the new algorithm. Fig. 5b shows the results of segmenting the same image by original algorithm with 1725 iterations taken to reach an essentially state since the path of curve evolution is worse. We also compared that the CPU time (in seconds) and the iterative times of the two algorithms, the results are given in Table I.
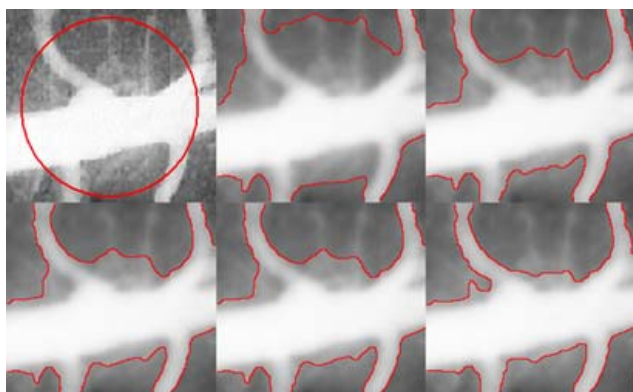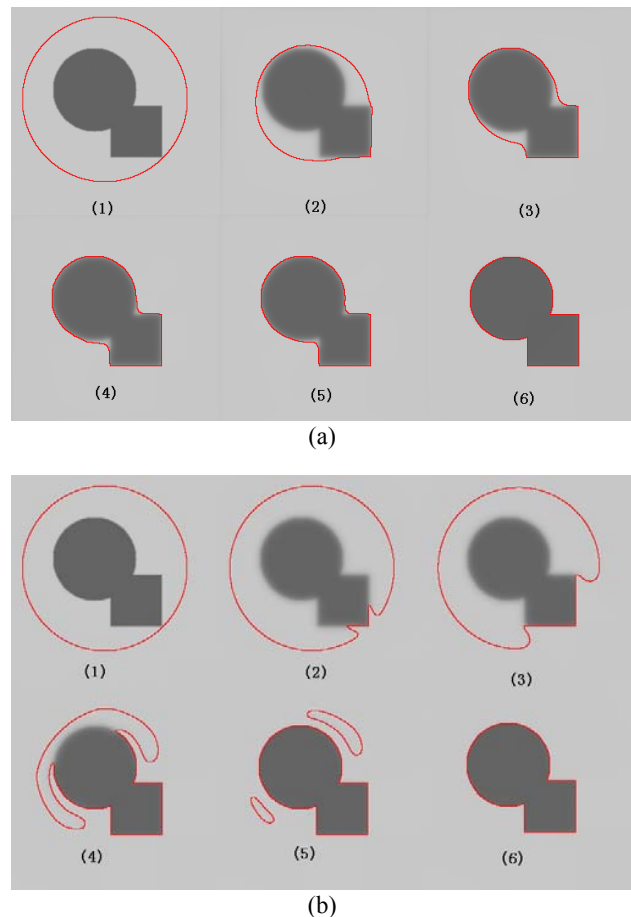

(a)

(b)

Fig. 5 Segmenting an artificial image with furry edges: (a) by the proposed algorithm and (b) by the original algorithm

Fig. 4 A noise image segmented by the proposed algorithm

1	2	3
4	5	6

TABLE I
COMPARISON OF THE PROPOSED ALGORITHM AND THE CLASSICAL ONE

Image	Method	Iterations	Time
blood vessel (Fig. 2)	Chan-Vese algorithm in the piecewise smooth case.	21500	8040s
	Proposed Method	15440	1320s
Strong noisy image (Fig. 3)	Chan-Vese algorithm in the piecewise smooth case.	27390	10240 s
	Proposed algorithm	20235	1730s
image with Gaussian noise (Fig. 4)	Proposed algorithm	19650	1680s
artificial image (Fig. 5)	Chan-Vese algorithm in the piecewise smooth case.	1725	640s
	Proposed algorithm	162	15s

V. CONCLUSION

One major advantage of the algorithm proposed in this paper is that it can be applied with all kinds of images, in which cases it still converges very rapidly. This is duo to the fact that the numerical solution for piecewise-constant MS segmentation model, which has been integrated to generate the additional force, is more efficient and accurate. For some high noise images, however, we have to limit the extra force at certain ranges. The tuning function, for simple, only is implemented based on the piecewise-smooth MS model and controlled by number of iterations in this paper. It is possible to find a better method for this purpose, which will be researched further.

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