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Automatic Iterative Methods for the Multivariate Solution of Nonlinear Algebraic Equations

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Abstract—Most real world systems express themselves formally as a set of nonlinear algebraic equations. As applications grow, the size and complexity of these equations also increase. In this work, we highlight the key concepts in using the homotopy analysis method as a methodology used to construct efficient iteration formulas for nonlinear equations solving. The proposed method is experimentally characterized according to a set of determined parameters which affect the systems. The experimental results show the potential and limitations of the new method and imply directions for future work.

Keywords—Nonlinear Algebraic Equations, Iterative Methods, Homotopy Analysis Method.

I. Introduction

A Problem encountered frequently in virtually any field of science, engineering, or applied mathematics is the solution of systems of nonlinear algebraic equations (SNAE). Moreover, SNAE are ubiquitous in the many applications requiring numerical simulation, and more robust and efficient methods for solving SNAE are continuously being sought.

Many powerful methods have been presented for the solutions of SNAE. So far, almost all iterative techniques require the prior one or more initial guesses for the desired root and may fail to converge when the initial guess in far from the required solution. Achieving convergence in an efficient manner in these situations has become a real challenge.

The Homotopy Analysis Method (HAM) is based on the classic homotopy theory and it is a general method for solving nonlinear problems. Liao proposed the method in 1992 and successively refined it [6-9]. A comprehensive treatment appears in [3-5]. Liao introduces a nonzero auxiliary parameter $\hbar$ to construct a new kind of homotopy:

$$\hat{H}(v; q, \hbar) = (1-q)\mathcal{L}[v(t; q, \hbar) - v_0(t)] - q\hbar\mathcal{A}[v(t; q, \hbar)].$$

Also, similar to the classical homotopy case, as q increases from 0 to 1, $v(t; q, \hbar)$ varies from the initial approximation $v_0(t)$ to the exact solution $v(t)$ of the original nonlinear model. However, in case of HAM, the solution $v(t; q, \hbar)$ of the equation $\hat{H}[v(t; q, \hbar)] = 0$ depends not only on the embedding parameter q, but also on the auxiliary parameter $\hbar$ and the linear operator $\mathcal{L}$. This provides us with a family of homotopy approximation series whose convergence region depends on the auxiliary parameter $\hbar$.

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The HAM method has demonstrated promise in the arena of analytical solutions of equations. Abbasbandy and et al. [1] have been successfully extended HAM to the iterative numerical solution of algebraic equations. Indeed the inherent flexibility and generality of the HAM method makes this a challenging task. The successful theoretical development of this methodology forms a core accomplishment of the work by Awawdeh [2]. Awawdeh developed the methodology for the iterative numeric solution of multivariate system of nonlinear algebraic equations.

In this work, we present HAM as a methodology used to construct efficient iterative numerical algorithms for the solutions of SNAE. Some problems were selected to illustrate the performance of our algorithms in solving SNAE.

II. Multivariate Solution of Nonlinear Algebraic Equations

Consider the system of nonlinear algebraic equations

$$F(x) = 0, \tag{1}$$

where $F(x) = (f_1(x), f_2(x), \cdots, f_n(x))^T$, $x = (x^{(1)}, x^{(2)}, \cdots, x^{(n)})^T$ and $f_i : \mathbb{R}^n \to \mathbb{R}$ is a nonlinear function. Assume that F has continuous second order partial derivatives on a convex set $D \subseteq \mathbb{R}^n$, and that it has a locally unique simple root $\alpha \in D$. Furthermore, assume that the Jacobian matrix $J[F(x)]$ is invertible in a neighborhood of α.

By using the truncated Taylor's expansion near x

$$F(x-\beta) = F(x) - J[F(x)]\beta + \frac{1}{2}H_\beta[F(x)], \tag{2}$$

where $\beta = [\beta_1, \beta_2, \cdots, \beta_n]^T$, $\beta_i = x_i - \alpha_i$ and

$$H_\beta[F(x)] \\ = [\beta^T H_1[F(x)]\beta, \beta^T H_2[F(x)]\beta, \ldots, \beta^T H_n[F(x)]\beta].$$

Here $H_i[F(x)]$ is the Hessian matrix of f_i

$$H_i[F(x)] = \begin{bmatrix} \frac{\partial^2 f_i}{\partial x_1^2} & \frac{\partial^2 f_i}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f_i}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f_i}{\partial x_2 \partial x_1} & \frac{\partial^2 f_i}{\partial x_2^2} & \cdots & \frac{\partial^2 f_i}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f_i}{\partial x_n \partial x_1} & \frac{\partial^2 f_i}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f_i}{\partial x_n^2} \end{bmatrix}.$$

We seek β such that $F(x-\beta) \approx 0$ and thus (2) gives

$$\beta = J^{-1}[F(x)]F(x) + \frac{1}{2}J^{-1}[F(x)]H_\beta[F(x)]. \tag{3}$$

Equation (3) is a nonlinear algebraic equation in β and can be decomposed into its linear and nonlinear constituents as follows

$$\mathcal{A}(\beta) = \mathcal{L}(\beta) + \mathcal{N}(\beta) = c, \tag{4}$$

where $\mathcal{L}(\beta) = \beta$ represent the linear component, $c = J^{-1}[F(x)]F(x)$ and

$$\mathcal{N}(\beta) = -\frac{1}{2}J^{-1}[F(x)]H_\beta[F(x)]$$

represents the nonlinear component.

We will apply the HAM to solve Equation (4) for β. Following the steps used in the previous section we have the series solution

$$\beta = v(1) \simeq \sum_{m=0}^{N} \beta_m \tag{5}$$

with the zero-order deformation equation

$$(1-q)\mathcal{L}[v(q) - \beta_0] = q\hbar(\mathcal{A}[v(q)] - c). \tag{6}$$

We will calculate the $Nth-$order approximation of β for various values of N in Equation (5).

Setting $\beta_0 = J^{-1}[F(x)]F(x)$ and taking the zeroth-order approximation ($N = 0$) of β in (5), we obtain

$$\alpha = x - \beta \simeq x - \beta_0 = x - J^{-1}[F(x)]F(x). \tag{7}$$

We can write the iteration form of (7) as follows

$$x_{n+1} = x_n - J^{-1}[F(x_n)]F(x_n),$$

which is the Newton-Raphson method.

Using first-order approximation ($N = 1$) of β in (5), we get

$$\beta = \beta_0 + \beta_1.$$

Note that β_1 can be obtained by differentiating (6) w.r.t q and keeping $q = 0$ to obtain the first order deformation equation

$$\begin{aligned} &-\mathcal{L}[v(0) - \beta_0] + (1-q)[\beta_1] \\ &= \hbar\{v(q) - \frac{1}{2}J^{-1}[F(x)]H_{\beta_0}[F(x)] - J^{-1}[F(x)]F(x)\}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} &H_{\beta_0}[F(x)] \\ &= [\beta_0^T H_1[F(x)]\beta_0, \beta_0^T H_2[F(x)]\beta_0, \ldots, \beta_0^T H_n[F(x)]\beta_0]. \end{aligned}$$

Plain calculations show that (8) reduces to

$$\beta_1 = -\frac{\hbar}{2}J^{-1}[F(x)]H_{\beta_0}[F(x)]$$

and so

$$\begin{aligned} \alpha = x - \beta \simeq x - \beta_0 = x - J^{-1}[F(x)]F(x) + \\ \frac{\hbar}{2}J^{-1}[F(x)]H_{\beta_0}[F(x)]. \end{aligned}$$

The iteration form of (8) can be given as follows

$$x_{n+1} = x_n - J^{-1}[F(x_n)]F(x_n) + \frac{\hbar}{2}J^{-1}[F(x_n)]H_{\beta_0}[F(x_n)]. \tag{9}$$

Awawdeh [2] showed that Iteration (9) can be written in operator form as follows: Given $x_0 \in D$, compute y_n, x_{n+1} from

$$0 = F(x_n) + F'(x_n)(y_n - x_n), \tag{10}$$

$$0 = F(x_n) + F'(x_n)(x_{n+1} - x_n) - \frac{\hbar}{2}F''(x_n)(y_n - x_n)^2,$$

$$n = 0, 1, 2, \ldots.$$

Notice that for $\hbar = -1$, HAM method (10) gives rise to

Chebyshev's method.

In actual computations the n^{th} step of (10) proceeds as follows:

1^{st} Stage: Compute a LR-decomposition of $F'(x_n)$ by Gaussian elimination.

2^{nd} Stage: Solve the linear system

$$F'(x_n)a_n = -F(x_n).$$

3^{rd} Stage: Solve the linear system

$$F'(x_n)b_n = -\frac{\hbar}{2}F''(x_n)a_n^2.$$

4^{th} Stage: Set $x_{n+1} = x_n + b_n$.

In the case $N = 2$, we have

$$\beta = \beta_0 + \beta_1 + \beta_2.$$

Differentiate (6) twice w.r.t q and set $q = 0$ to obtain the second order deformation equation

$$\begin{aligned} 2(\beta_2 - \beta_1) = 2\hbar\{\beta_1 - \frac{1}{2}J^{-1}[F(x)]H_a[F(x)] - \\ \frac{1}{2}J^{-1}[F(x)]H_b[F(x)]\} \end{aligned} \tag{11}$$

where

$$\begin{aligned} &H_a[F(x)] \\ &= [\beta_1^T H_1[F(x)]\beta_0, \beta_1^T H_2[F(x)]\beta_0, \ldots, \beta_1^T H_n[F(x)]\beta_0] \end{aligned}$$

and

$$\begin{aligned} &H_b[F(x)] \\ &= [\beta_0^T H_1[F(x)]\beta_1, \beta_0^T H_2[F(x)]\beta_1, \ldots, \beta_0^T H_n[F(x)]\beta_1]. \end{aligned}$$

Note that $H_a[F(x)] = H_b[F(x)]$ since the Hessian matrix is symmetrical. Equation (11) therefore reduces to

$$\beta_2 = \beta_1 + \hbar\{\beta_1 - J^{-1}[F(x)]H_a[F(x)]\}$$

and we get

$$\begin{aligned} \alpha = x - \beta \simeq x - J^{-1}[F(x)]F(x) + \\ (2+\hbar)\frac{\hbar}{2}J^{-1}[F(x)]H_{\beta_0}[F(x)] + \hbar J^{-1}[F(x)]H_a[F(x)]. \end{aligned} \tag{12}$$

The iteration form of (12) can be given as follows

$$\begin{aligned} x_{n+1} = x_n - J^{-1}[F(x_n)]F(x_n) + \\ (2+\hbar)\frac{\hbar}{2}J^{-1}[F(x_n)]H_{\beta_0}[F(x_n)] + \\ \hbar J^{-1}[F(x_n)]H_a[F(x_n)]. \end{aligned} \tag{13}$$

III. Applications

In this section, we have selected some examples which will show the simplicity and effectiveness of the proposed algorithms. The calculations were done using MATLAB 7. Our comparison of the methods is based upon the number of iterations. We use the following stopping criterion for our computer programs:

$$|x_{k+1} - x_k| < \epsilon,$$

where $\epsilon = 2.22 \times 10^{-16}$ is a MATLAB constant.

Example 1: Consider the system of equations

$$\begin{aligned} 0 &= x^2 - 10x + y^2 + 8, \\ 0 &= xy^2 + x - 10y + 8, \end{aligned}$$

which has the roots $\alpha_1 = (1, 1)$ and $\alpha_2 = (2.193, 3.020)$. The maximum number of iterations using (13) in the range of $\hbar = (-1, 1)$, where $\hbar$ is varied uniformly in this range with a step size of 0.002 and $x_0 = (-2, 2)$, is 6.

Example 2: Consider the system of three nonlinear equations in three unknowns:

$$\begin{aligned} 0 &= 3x - \cos(yz) - 0.5, \\ 0 &= x^2 - 81(y + 0.1)^2 + \sin z + 1.06, \\ 0 &= e^{-xy} + 20z + \frac{10\pi - 3}{3}. \end{aligned}$$

This has a zero

$$\alpha \simeq (0.49814468, -0.19960589, -0.52882597).$$

For $x_0 = (5, 5, 2)$ the proper values of $\hbar$ are $\hbar \in (-1, -0.55)$ and for $x_0 = (-15, -15, -15)$ are $\hbar \in (0, 0.018)$. Table I illustrates that if x_0 is not sufficiently close to the actual root, there is enough reason to suspect that Newton's method (NM), Chebyshev's method (CM) and Halley's method (HM) will diverge and in this case we can still find a proper value of $\hbar$ that ensures the convergence of method (10).

TABLE I
Numerical results of the solutions in Example 4

$method$	$x_0 = (5, 5, 2)$	$x_0 = (-15, -15, -15)$
	$iter$	$iter$
NM	14	$Divergent$
CM	8	$Divergent$
HM	8	$Divergent$
HAM	$6(\hbar = -0.6)$	$9(\hbar = 0.001)$

IV. Conclusion

The HAM algorithms are very effective and efficient which provide highly accurate results in less number of iterations as compared to some well-known existing methods. It is shown to have significant advantages over the traditional methods in terms of flexibility, convergence and possibly speed. One of the disadvantages of the algorithms that it require the computation of high derivatives. But, The practical relevance of these methods increases since computer aided formulae manipulation facilities became a common tool in numerical analysis. HAM algorithms contain the parameter $\hbar$ which can be used to ensure and accelerate the convergence. In future work, we will seek an efficient method to get optimal values of $\hbar$.

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