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Numerical Algorithms for Solving a Type of Nonlinear Integro-Differential Equations

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***Abstract*—In this article two algorithms, one based on variation iteration method and the other on Adomian's decomposition method, are developed to find the numerical solution of an initial value problem involving the nonlinear integro-differential equation

$$\frac{\partial}{\partial t}u(x,t)+\int_0^t Ru(x,s)ds=g(x,t)$$

where R is a nonlinear operator that contains partial derivatives with respect to x. Special cases of the integro-differential equation are solved using the algorithms. The numerical solutions are compared with analytical solutions. The results show that these two methods are efficient and accurate with only two or three iterations

Keywords—variation iteration method, decomposition method, nonlinear integro-differential equations

I. Introduction

The equation

$$\frac{\partial}{\partial t}u(x,t)+\int_0^t Ru(x,s)ds=g(x,t) \qquad (1)$$

is an example of general nonlinear integro-differential equations defined on a Hilbert space. In the equation R is a nonlinear operator that contains partial derivatives with respect to x, and g is an inhomogeneous term. Of particular interest is the following special case

$$\frac{\partial}{\partial t}u(x,t)-\int_0^t a(t-s)\frac{\partial}{\partial x}\sigma\left(\frac{\partial}{\partial x}u(x,s)\right)ds=g(x,t),$$

$$0<x<1, \qquad 0<t<T. \qquad (2)$$

with the initial condition

$$u(x,0)=f(x) \qquad (3)$$

The problem arises in the theory of one-dimensional viscoelasticity [8] – [10]. It is also a special model for one-dimensional heat flow in materials with memory [5].

A numerical solution to the nonlinear problem given by (2) and (3) was obtained using Galerkin's method [11]. In this paper, the variation iteration method and decomposition method are described and applied to compute numerical solutions to (2) and (3). It will be shown that the algorithms are efficient and accurate with only two or three iterations.

The article has been organized as follows: In Section 2, the application of variation iteration method to solve the nonlinear problem (2) and (3) is discussed; in Section 3, an improved decomposition algorithm is presented to find numerical solutions; and in Section 4 the two algorithms are applied to examples, and the numerical results with analytical solutions are compared.

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II. The Variation Iteration Method

The variation iteration method (VIM) [6]–[7] was proposed by He to solve nonlinear differential equations using an iterative formula. In this section a VIM algorithm is developed to solve the nonlinear integro-differential equations (1)

Applying the variation iteration method to (1), we construct the following iteration formula:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\tau)\left[\frac{\partial}{\partial\tau}u_n(x,\tau)+\int_0^\tau R\tilde{u}_n(x,s)ds-g(x,\tau)\right]d\tau \qquad (4)$$

where λ is a general Lagrangian multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is considered as a restricted variation [7], that is, $\delta\tilde{u}_n=0$.

By taking variation with respect to u_n and noticing that $\delta R\tilde{u}_n=0$ it can be derived that

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda(\tau)\left[\frac{\partial}{\partial\tau}u_n(x,\tau)+\int_0^\tau R\tilde{u}_n(x,s)ds-g(x,\tau)\right]d\tau$$

$$=\delta u_n(x,t)+\lambda(\tau)\delta u_n(x,\tau)|_{\tau=t}-\int_0^t \lambda'(\tau)\delta u_n(x,\tau)\,d\tau=0.$$

This yields the stationary conditions:

$$\lambda'(\tau)=0, \text{ and } 1+\lambda(\tau)|_{\tau=t}=0.$$

Therefore, the Lagrangian multiplier $\lambda(\tau)=-1$.

Substituting the identified multiplier into (4) the following iteration formula is obtained:

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left[\frac{\partial}{\partial\tau}u_n(x,\tau)+\int_0^\tau Ru_n(x,s)ds-g(x,\tau)\right]d\tau, \qquad (5)$$

for $n\ge 0$ with $u_0(x,t)$ chosen to be $u_0(x,t)=u(x,0)=f(x)$.

Integration by parts yields

$$\int_0^t \frac{\partial}{\partial\tau}u_n(x,\tau)d\tau=u_n(x,t)-u_n(x,0)$$

and, therefore, a simpler version of the iteration formula (5) can be obtained

$$u_{n+1}(x,t)=u_n(x,0)-\int_0^t\left[\int_0^\tau Ru_n(x,s)ds-g(x,\tau)\right]d\tau, \qquad n\ge 0 \qquad (6)$$

To further simplify the iteration formula, it is observed that

$$u_1(x,0)=u_0(x,0)-\int_0^0\left[\int_0^\tau Ru_0(x,s)ds-g(x,\tau)\right]d\tau=u_0(x,0)=u(x,0)$$

$$u_2(x,0)=u_1(x,0)$$

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$$-\int_0^0\left[\int_0^\tau Ru_1(x,s)ds-g(x,\tau)\right]\mathrm{d}\tau=u_1(x,0)=u(x,0)$$
$$\vdots$$

It implies by induction that $u_n(x,0)=u(x,0)$ for any positive integer *n*, and therefore,

$$u_{n+1}(x,t)=u(x,0)-\int_0^{\mathrm{t}}\left[\int_0^\tau Ru_n(x,s)ds-g(x,\tau)\right]\mathrm{d}\tau,\qquad n\geq 0 \quad (7)$$

Applying iteration formula (7) to (2) with initial condition (3) leads to the following theorem.

Theorem 2.1 The solution to the integro-differential equation (2) with initial condition (3) can be determined by the iteration formula

$$u_{n+1}(x,t)=f(x)+\int_0^t\left[\int_0^\tau a(\tau-s)\frac{\partial}{\partial x}\sigma\left(\frac{\partial}{\partial x}u_n(x,s)\right)ds+g(x,\tau)\right]d\tau$$

III. The Decomposition Method

In this section an algorithm based on the Adomian's decomposition method (ADM) [1], [2], [4] is developed to solve the integro-differential equation (1):

$$\frac{\partial}{\partial t}u(x,t)+\int_0^t Ru(x,s)ds=g(x,t)$$

with the initial condition (3): $u(x,0)=f(x)$.

By the decomposition algorithm, a series expansion is assumed for u given by

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots. \quad (8)$$

Integrate both sides of (1) from 0 to t to obtain

$$u(x,t)-u(x,0)+\int_0^t\int_0^\tau Ru(x,s)dsd\tau=\int_0^t g(x,\tau)d\tau \quad (9)$$

Applying (9) to the integro-differential equation (2) with the initial condition (3) yields

$$u(x,t)-u(x,0)-\int_0^t\int_0^\tau a(\tau-s)\frac{\partial}{\partial x}\sigma\left(\frac{\partial}{\partial x}u(x,s)\right)dsd\tau=\int_0^t g(x,t)d\tau \quad (10)$$

In series (8) let $u_0(x,t)=u(x,0)+\int_0^t g(x,\tau)d\tau$, and thus it follows from (10) and (8) that

$$\begin{aligned}
&u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots\\
&=\int_0^t\int_0^\tau a(\tau-s)\frac{\partial}{\partial x}\sigma\left[\frac{\partial}{\partial x}u_0(x,s)+\frac{\partial}{\partial x}u_1(x,s)+\frac{\partial}{\partial x}u_2(x,s)+\cdots\right]dsd\tau\\
&=\int_0^t\int_0^\tau a(\tau-s)\frac{\partial}{\partial x}[B_0(x,s)+B_1(x,s)+B_2(x,s)+\cdots]dsd\tau
\end{aligned}$$

where

$$\sum_{i=0}^{\infty}B_i(x,s)=\sigma\left[\frac{\partial}{\partial x}u_0(x,s)+\frac{\partial}{\partial x}u_1(x,s)+\frac{\partial}{\partial x}u_2(x,s)+\cdots\right] \quad (11)$$

Note that B_i 's $(i=0,1,2,\cdots)$ are specially generated decomposition polynomials that depend only on components from $\frac{\partial u_0}{\partial x},\frac{\partial u_1}{\partial x},\cdots$, to $\frac{\partial u_i}{\partial x}$. To be more specific, we define the order of the component $\left(\frac{\partial u_j}{\partial x}\right)^m$ to be jm, and $\left(\frac{\partial u_j}{\partial x}\right)^m\cdot\left(\frac{\partial u_k}{\partial x}\right)^n$ to be $jm+kn$. Then the decomposition polynomial B_0 depends upon the components with order 0, B_1 depends upon components with order 1, etc..

A special case $\sigma(\xi)=\xi^2$ can be used to better explain the construction of the decomposition polynomial B_i. It is derived from (11) that

$$\begin{aligned}
&B_0(x,s)+B_1(x,s)+B_2(x,s)+\cdots\\
&=\sigma\left[\frac{\partial u_0}{\partial x}(x,s)+\frac{\partial u_1}{\partial x}(x,s)+\frac{\partial u_2}{\partial x}(x,s)+\cdots\right]\\
&=\left[\frac{\partial u_0}{\partial x}(x,s)+\frac{\partial u_1}{\partial x}(x,s)+\frac{\partial u_2}{\partial x}(x,s)+\cdots\right]^2\\
&=\left(\frac{\partial u_0}{\partial x}\right)^2+2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}\right)+2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_2}{\partial x}\right)\\
&\quad+\left(\frac{\partial u_1}{\partial x}\right)^2+2\left(\frac{\partial u_1}{\partial x}\right)\left(\frac{\partial u_2}{\partial x}\right)+2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_3}{\partial x}\right)+\cdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
B_0&=\left(\frac{\partial u_0}{\partial x}\right)^2\\
B_1&=2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}\right)\\
B_2&=2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_2}{\partial x}\right)+\left(\frac{\partial u_1}{\partial x}\right)^2\\
B_3&=2\left[\left(\frac{\partial u_1}{\partial x}\right)\left(\frac{\partial u_2}{\partial x}\right)+\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_3}{\partial x}\right)\right] \qquad (12)\\
&\vdots
\end{aligned}$$

Notice that B_0 is determined by the component $\left(\frac{\partial u_0}{\partial x}\right)^2$ of order 0, B_1 is determined by the component $\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}\right)$ of order 1, ⋯, etc..

The above analysis leads to the following theorem:

Theorem 3.1 The solution to the integro-differential equation (2) and (3) can be determined by the series (8) with u_i $(i=0,1,2,\cdots)$ given by the iterations

$$\begin{aligned}
u_0(x,t)&=u(x,0)+\int_0^t g(x,\tau)d\tau\\
u_1(x,t)&=\int_0^t\int_0^\tau a(\tau-s)\frac{\partial}{\partial x}B_0(x,s)dsd\tau,\\
u_2(x,t)&=\int_0^t\int_0^\tau a(\tau-s)\frac{\partial}{\partial x}B_1(x,s)dsd\tau,\\
&\vdots\\
u_n(x,t)&=\int_0^t\int_0^\tau a(\tau-s)\frac{\partial}{\partial x}B_{n-1}(x,s)dsd\tau
\end{aligned}$$

for $n=1,2,\cdots$, where B_i $(i=0,1,2,\cdots)$ are the terms of expansion (11).

For the series solution in decomposition algorithm to converge, two hypotheses are needed, [3]: The nonlinear equation has a series solution such that $\sum_{n=0}^{\infty}(1+\epsilon)^n|u_n|<\infty$ for small ϵ, and the nonlinear operator R can be developed in series $R(u)=\sum_{n=0}^{\infty}\alpha_n u^n$. These two hypotheses are usually satisfied in many physical problems.

IV. Examples

In this section different forms of the kernel $a(\cdot)$ and the nonlinear function $\sigma(\cdot)$ [11] in (2) are considered. The inhomogeneous term $g(x,t)$ and initial condition $f(x)$ in (3) are also chosen appropriately so that exact solutions are available. The exact solutions are then compared with the numerical solutions derived through the variation iteration method and decomposition algorithm.

Example 1: In this example, $a(\xi)=e^{-\xi}$, $\sigma(\xi)=\xi^2$,

$$g(x,t)=e^{-(x+t)}+2e^{-2x}(e^{-t}-e^{-2t}),$$

and the initial condition $u(x,0)=e^{-x}$. With these choices, (2) and (3) become

$$\frac{\partial}{\partial t}u(x,t)-\int_0^t e^{-(t-s)}\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x}u(x,s)\right)^2\right]ds = e^{-(x+t)}+2e^{-2x}(e^{-t}-e^{-2t}),$$

$$u(x,0)=e^{-x}.$$

The exact solution for this problem is $u(x,t)=e^{-(x+t)}$.

The variation iteration formula in Theorem 2.1 for the example takes the form

$$u_{n+1}(x,t)=e^{-x}+\int_0^t\int_0^\tau e^{-(\tau-s)}\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x}u_n(x,s)\right)^2\right]ds\,d\tau+\int_0^t\left[e^{-(x+\tau)}+2e^{-2x}(e^{-\tau}-e^{-2\tau})\right]d\tau \tag{13}$$

with $u_0(x,t)=e^{-x}$.

To find the series solution (8) of the decomposition method, the first terms $u_i(x,t)$ $(i=0,1,2,3)$ in Theorem 3.1 are expressed in terms of the decomposition polynomial B_i's in (12).

$$u_0(x,t)=e^{-x}+\int_0^t\left[e^{-(x+\tau)}+2e^{-2x}(e^{-\tau}-e^{-2\tau})\right]d\tau$$

$$u_1(x,t)=\int_0^t\int_0^\tau e^{-(\tau-s)}\frac{\partial}{\partial x}\left[\left(\frac{\partial u_0}{\partial x}\right)^2\right]ds\,d\tau,$$

$$u_2(x,t)=\int_0^t\int_0^\tau e^{-(\tau-s)}\frac{\partial}{\partial x}\left[2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}\right)\right]ds\,d\tau,$$

$$u_3(x,t)=\int_0^t\int_0^\tau e^{-(\tau-s)}\frac{\partial}{\partial x}\left[2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_2}{\partial x}\right)+\left(\frac{\partial u_1}{\partial x}\right)^2\right]ds d\tau, \tag{14}$$

$$\vdots$$

The series solution (8) of the problem (2) and (3) can then be approximated by

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots$$

Table 1 shows the errors between the exact solution and numerical solutions. The numerical error $|u_{Exact}-u_{ADM}|$ results from using three terms of the decomposition method, and the error $|u_{Exact}-u_{VIM}|$ from two iterations of variation iteration method both computed at $t=0.01$ and $x=0.0$, 0.2, ⋯, 1.0.

TABLE I
Error Comparison for Example 1

x	$\lvert u_{Exact}-u_{VIM}\rvert$	$\lvert u_{Exact}-u_{ADM}\rvert$
0.0	1.9898e-02	1.9928e-02
0.2	1.6291e-02	1.6296e-02
0.4	1.3339e-02	1.3339e-02
0.6	1.0921e-02	1.0921e-02
0.8	8.9413e-03	8.9415e-03
1.0	7.3206e-03	7.3207e-03

The table indicates that the two methods both have very reasonable accuracy. Furthermore, Figs. 1 and 2 also show good agreement between the graphs of the exact solution and those of the numerical solutions of VIM and ADM.

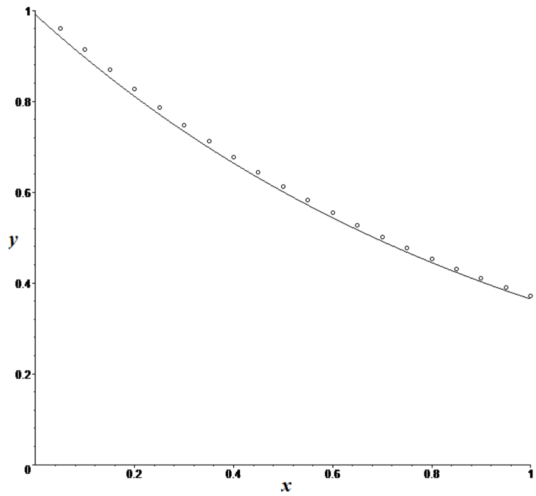

Fig. 1 Exact solution (curve) vs VIM solution (discrete circles)

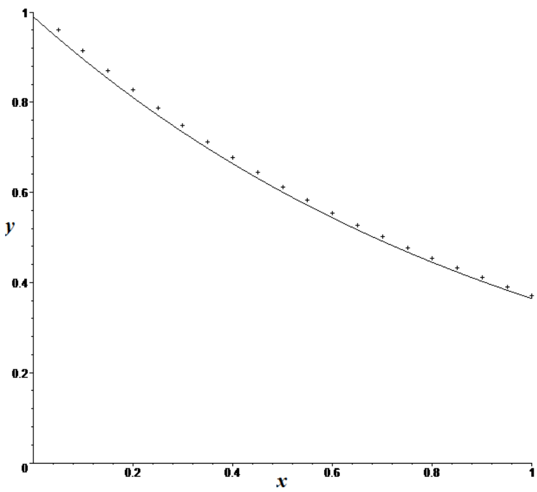

Fig. 2 Exact solution (curve) vs ADM solution (discrete crosses)

Example 2: In this example we choose $a(\xi)=e^{-2\xi}$, $\sigma(\xi)=\xi^2$, and $g(x,t)=\cos(x+t)+¼[\sin 2(x+t)-\cos 2(x+t)-e^{-2t}(\sin 2x-\cos 2x)]$ in (2), and let the initial condition $u(x,0)=\sin x$.

The exact solution is $u(x,t)=\sin(x+t)$ for these choices.

The variation iteration formula in Theorem 2.1 for the example takes the form

$$u_{n+1}(x,t) = \sin x + \int_0^t \int_0^\tau e^{-2(\tau-s)} \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x}u(x,s)\right)^2\right] ds d\tau$$
$$+ \int_0^t \{\cos(x+\tau) + ¼[\sin 2(x+\tau) - \cos 2(x+\tau)$$
$$-e^{-2\tau}(\sin 2x - \cos 2x)]\} d\tau \qquad (15)$$

with $u_0(x,t) = \sin x$.

The u_i's in the decomposition method can be determined by

$$u_0(x,t) = \sin x + \int_0^t \{\cos(x+\tau) + ¼[\sin 2(x+\tau)$$
$$-\cos 2(x+\tau) - e^{-2\tau}(\sin 2x - \cos 2x)]\} d\tau$$
$$u_1(x,t) = \int_0^t \int_0^\tau e^{-2(\tau-s)} \frac{\partial}{\partial x}\left[\left(\frac{\partial u_0}{\partial x}\right)^2\right] ds\, d\tau$$
$$u_2(x,t) = \int_0^t \int_0^\tau e^{-2(\tau-s)} \frac{\partial}{\partial x}\left[2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}\right)\right] ds\, d\tau$$
$$\vdots \qquad (16)$$

We let $t = 0.3$, and use three iterations of variation iteration method and two terms of decomposition approximation to compute the numerical solutions of the problems (2) and (3). The errors of these solutions compared with the exact solution are listed in Table 2.

In the table very reasonable accuracy is reached by both methods. When x approaches 1.0, the decomposition method is slightly better than the variation iteration method. We can also observe this phenomenon as we compare the graphs in Figs. 3 and 4.

TABLE II
ERROR COMPARISON FOR EXAMPLE 1

x	$\lvert u_{Exact} - u_{VIM}\rvert$	$\lvert u_{Exact} - u_{ADM}\rvert$
0.0	6.7232e-03	4.5787e-04
0.2	3.5375e-03	1.3564e-03
0.4	1.3665e-04	1.7970e-03
0.6	4.2055e-03	1.6572e-03
0.8	8.3884e-03	1.0200e-03
1.0	1.2062e-03	1.4095e-04

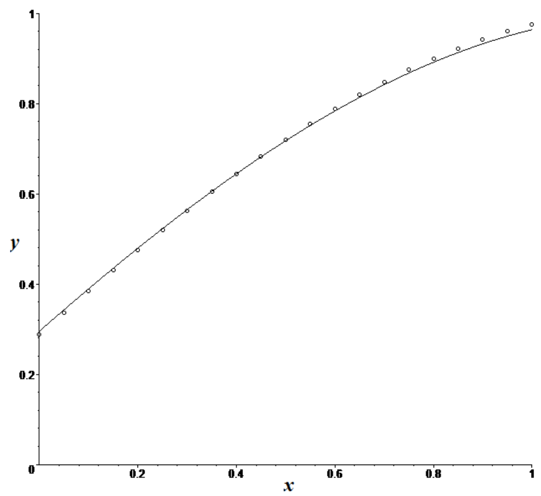

Fig. 3 Exact solution (curve) vs VIM solution (discrete circles)

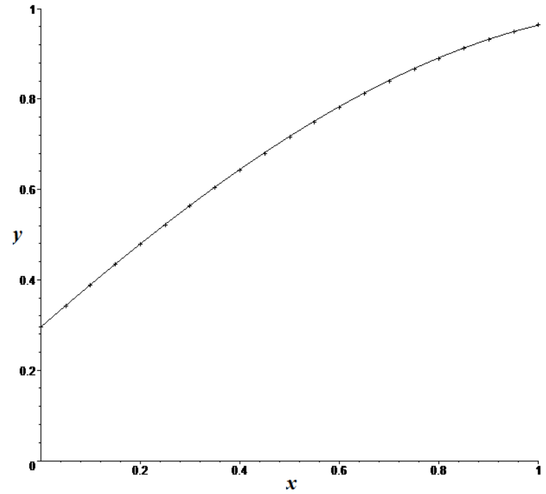

Fig. 4 Exact solution (curve) vs ADM solution (discrete crosses)

V. CONCLUSION

The variation iteration and decomposition algorithms are both capable of solving nonlinear equations. Indeed, the examples show that the error between the exact solution and the numerical solutions obtained by these two algorithms are small, and this was achieved using only two or three iterations. It should also be remarked that the graphs drawn using VIM and ADM are also in good agreement with those of the exact solutions. The two methods involve reasonable amount of computations, which is handled by Maple.

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