Analytical Solution for Compressible Gas Flow Inside a Two-Dimensional Poiseuille Flow in Microchannels with Constant Heat Flux Including the Creeping Effect

Amir Reza Ghahremani, Salman SafariMohsenabad, and Mohammad Behshad Shafii

Abstract—To achieve reliable solutions, today's numerical and experimental activities need developing more accurate methods and utilizing expensive facilities, respectfully in microchannels. The analytical study can be considered as an alternative approach to alleviate the preceding difficulties. Among the analytical solutions, those with high robustness and low complexities are certainly more attractive. The perturbation theory has been used by many researchers to analyze microflows. In present work, a compressible microflow with constant heat flux boundary condition is analyzed. The flow is assumed to be fully developed and steady. The Mach and Reynolds numbers are also assumed to be very small. For this case, the creeping phenomenon may have some effect on the velocity profile. To achieve robustness solution it is assumed that the flow is quasi-isothermal. In this study, the creeping term which appears in the slip boundary condition is formulated by different mathematical formulas. The difference between this work and the previous ones is that the creeping term is taken into account and presented in non-dimensionalized form. The results obtained from perturbation theory are presented based on four non-dimensionalized parameters including the Reynolds, Mach, Prandtl and Brinkman numbers. The axial velocity, normal velocity and pressure profiles are obtained. Solutions for velocities and pressure for two cases with different Br numbers are compared with each other and the results show that the effect of creeping phenomenon on the velocity profile becomes more important when Br number is less than $O(\epsilon)$.

Keywords—Creeping Effect, Microflow, Slip, Perturbation.

I. Nomenclature

Br	Brinkman number, Eq. (3)
Kn	Knudsen number, Eq. (3)
Pr	Prandtl number
M	Mach number, Eq. (3)
Re	Reynolds number, Eq. (3)
H, L	microchannel height and length
T	temperature
u, ν	velocity
γ	specific heat ratio
ϵ	height to length ratio, Eq. (3)
σ_ν	momentum accommodation coefficient
λ	mean free path
μ	gas viscosity
ρ	density

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A. Subscripts

i, o	averaged inlet and outlet property
w	wall property

B. Superscript

$-$	nondimensional property

II. Introduction

Microchannels are important components for many micro-electro-mechanical systems (MEMS). From a microflow perspective, the micro-electro-mechanical systems (MEMS) are devices with a characteristic length of less than 1 mm but more than 1 μm; therefore, the flow Knudsen number in such devices is characterized between 0.01 and 0.1, which is in the range of a slip flow regime. The correct slip-velocity boundary condition implementation is a rather important task in simulating slip flow regime. Because of rarefaction effects, the no-slip boundary condition cannot be implemented anymore on solid boundaries [1]. For many years, the micro flow through micro tubes and microchannels has been investigated experimentally, numerically, and analytically. Numerically, there were many simulations of compressible flows in a microchannel as well. Examples are the direct simulation Monte Carlo method [2−4], the Information Preservation method [5], the direct-solving Boltzmann method [6], the Boltzmann equations [7], and gas-kinetic BGKBurnett method [8]. Discussions of the thermal heating are also reported as well [9,10]. There are some analytical works which solve Navier-Stokes equations for both compressible and incompressible flows. Tunc and Bayazitoglu [11] analytically studied the flow in microtubes. They used the method of separation of variables to obtain the temperature profile imposing constant wall temperature condition. Xu et al. [12] theoretically analyzed and examined the effects of viscous dissipation in microchannel flows. They suggested a criterion to determine the limits of viscous dissipation significance. The past studies show that the Navier-Stokes equations can be safely used to derive slip flow solutions in this range, e.g. see Darbandi and Vakilipour [13,14]. One of the most familiar analytical methods, which is used to analyze compressible flows, is the perturbation theory. We consider the effect of the creeping phenomenon and demonstrate its term in the slip boundary condition in non-dimensionalized forms. The
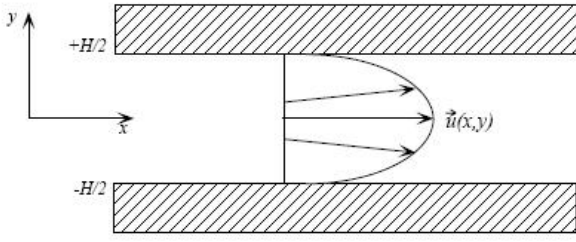
Fig. 1. Geometry for channel analysis, with a flow profile at a given position

creeping effect is expressed in non-dimensionalized numbers including the Br, Re, and M. The first order analytical solution is obtained by using the perturbation theory.

III. RELATED WORK

In a recent attempt, Cai and Boyd [15] used the perturbation theory and analyzed compressible gas flow through a microchannel. They presented a complete set of the first-order analytical solutions for pressure and velocity profiles. Their work is an extension of the original work of Arkilic et. al. [16], who conducted studies in compressible flows through microchannels. The present paper is the extension of the work presented by Cai [15] including the creeping phenomenon which is imposed through a boundary condition.

IV. ANALYSIS

In this section the effect of the creeping phenomenon on the velocity profile is demonstrated. The flow is assumed to be compressible and fully developed micro-Poiseuille. Also, constant heat flux at the wall is imposed as a boundary condition. Fig.(1) shows the geometry and a velocity profile at a given position of the microchannel used for this analysis. The continuity equation and the axial momentum equation can be written as

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho \nu)}{\partial y} = 0 \tag{1}$$

and

$$\rho\{u\frac{\partial u}{\partial x} + \nu\frac{\partial \nu}{\partial y}\} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\{\mu(2\frac{\partial u}{\partial x} - \frac{2}{3}(\frac{\partial u}{\partial x} + \frac{\partial \nu}{\partial y}))\}$$
$$+ \frac{\partial}{\partial y}\{\mu(\frac{\partial u}{\partial y} + \frac{\partial \nu}{\partial x})\} \tag{2}$$

, respectively. To simplify above equations, some non-dimensional variables introduced as follows

$$\epsilon = \frac{H}{L}, Kn = \frac{\lambda}{H}, Re = \frac{\rho u_o H}{\mu},$$

$$M = \frac{u_o}{\sqrt{\gamma R T_o}}, Br = \frac{\mu u_o^2}{(\frac{H}{2})q_s''}, \overline{x} = \frac{x}{L},$$

$$\overline{y} = \frac{y}{H}, \overline{P} = \frac{P}{P_{out}}, \overline{u} = \frac{u}{u_{out}}, \overline{\rho} = \frac{\rho}{\rho_{out}}. \tag{3}$$

Since the creeping phenomenon is important for low velocity flows (Cai et. al. [15]), it is assumed that $Re \sim O(\epsilon)$,

and $M \sim O(\epsilon)$ and $Kn \sim O(1)$. It is evident that these assumptions, which was adopted by Arkilic [16], is a special case from this category. According to the Schaaf et. al. [17], the slip velocity can be expressed in the form of

$$u_s = \frac{2 - \sigma_\nu}{\sigma_\nu}\lambda\frac{\partial u}{\partial y}|_{y=\frac{H}{2}} + \frac{3}{4}\frac{Pr(\gamma - 1)}{\gamma\rho R T_w}(q_s'') \tag{4}$$

By using defined parameters the Eq.(1) and Eq.(2) can be non-dimensionalized to yield

$$\epsilon\frac{\partial(\overline{\rho}\ \overline{u})}{\partial\overline{x}} + \frac{\partial(\overline{\rho}\ \overline{\nu})}{\partial\overline{y}} = 0 \tag{5}$$

and

$$Re\overline{P}(\epsilon\overline{u}\frac{\partial\overline{u}}{\partial\overline{x}} + \overline{\nu}\frac{\partial\overline{u}}{\partial\overline{y}}) = -\frac{\epsilon Re}{\gamma M^2}\frac{\partial\overline{P}}{\partial\overline{x}} + \epsilon^2\frac{\partial^2\overline{u}}{\partial\overline{x}^2} + \frac{\partial^2\overline{u}}{\partial\overline{y}^2}$$
$$+ \frac{1}{3}(\epsilon^2\frac{\partial^2\overline{u}}{\partial\overline{x}^2} + \epsilon\frac{\partial^2\overline{\nu}}{\partial\overline{x}\partial\overline{y}}) \tag{6}$$

, respectively. Also, the non-dimensional boundary conditions are

$$\overline{u}_s = \frac{2 - \sigma_\nu}{\sigma_\nu}Kn\frac{\partial\overline{u}}{\partial\overline{y}}|_{\overline{y}=0.5} + \frac{3}{2}\frac{Pr(\gamma - 1)M^2}{Br.Re} \tag{7}$$

$$\frac{\partial\overline{u}}{\partial\overline{y}}|_{\overline{y}=0} = 0 \tag{8}$$

To solve governing equations analytically using the perturbation theory, following formats are assumed for the non-dimensional quantities [16]

$$\overline{P} = \overline{P_o} + \epsilon\overline{P_1} + \epsilon^2\overline{P_2} + O(\epsilon^3)$$

$$\overline{u} = \overline{u_o} + \epsilon\overline{u_1} + \epsilon^2\overline{u_2} + O(\epsilon^3)$$

$$\overline{\nu} = \overline{\nu_o} + \epsilon\overline{\nu_1} + \epsilon^2\overline{\nu_2} + O(\epsilon^3) \tag{9}$$

Expanding and separating different magnitudes in Eq. (5), with nonporous wall condition, results in $\overline{\nu_o} = 0$. According to Cai and Boyd [15] it can be proven that $\overline{P_o} = P(x)$. Considering $O(1)$ for Eq. (6), The following relation for $\overline{u_o}$ can be obtained

$$O(1): -\frac{\epsilon Re}{\gamma M^2}\frac{\partial\overline{P_o}}{\partial\overline{x}} + \frac{\partial^2\overline{u}_o}{\partial\overline{y}^2} = 0 \tag{10}$$

Integrating Eq. (10) twice with respect to $\overline{y}$ leads to Eq. (11).

$$\overline{u_o} = \frac{\epsilon Re}{2\gamma M^2}\frac{\partial\overline{P_o}}{\partial\overline{x}}\overline{y}^2 + c_1\overline{y} + c_2 \tag{11}$$

Where c_1 and c_2 are integral constants. Using appropriate boundary conditions, one can find c_1 and c_2.

A. First Case

If $Br \sim O(\epsilon)$ then from Eq. (7) it can be concluded that

$$O(1): \overline{u}_{0s} = \frac{2 - \sigma_\nu}{\sigma_\nu}Kn\frac{\partial\overline{u}_o}{\partial\overline{y}}|_{\overline{y}=0.5} + \frac{3}{2}\frac{Pr(\gamma - 1)M^2}{Br.Re} \tag{12}$$

$$\frac{\partial\overline{u_0}}{\partial\overline{y}}|_{\overline{y}=0} = 0 \tag{13}$$

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The integral constants c_1 and c_2 of the Eq. (11) can be determined from boundary conditions Eq. (12) and Eq. (13). Therefore,

$$\overline{u}_o(\overline{x}, \overline{y}) = \frac{-\epsilon Re}{8\gamma M^2} \frac{\partial \overline{p}_o}{\partial \overline{x}} \{1 - 4\overline{y}^2 + 4(\frac{2-\sigma_\nu}{\sigma_\nu})Kn\}$$

$$+ \frac{3}{2} \frac{Pr(\gamma-1)M^2}{Br.Re} \quad (14)$$

Solving Eq. (5) for $\overline{\nu_1}$ and considering, $\overline{\nu_1} = 0$ on the wall surface as the boundary condition, $\overline{\nu_1}$ can be expressed as

$$\overline{\nu_1} = \frac{\epsilon Re}{8\gamma M^2} \frac{1}{\overline{P}_o} [\frac{1}{2}(\frac{\partial^2 \overline{P}_o^2}{\partial \overline{x}^2})(\overline{y} - \frac{4}{3}\overline{y}^3)$$

$$+ 4Kn_o(\frac{2-\sigma_\nu}{\sigma_\nu})\frac{\partial^2 \overline{P}_o^2}{\partial \overline{x}^2}\overline{y}] - \frac{3}{2}\frac{Pr(\gamma-1)M^2}{Br.Re}\frac{1}{\overline{P}_o}\frac{\partial \overline{P}_o}{\partial \overline{x}}\overline{y} \quad (15)$$

Evaluating the preceding equation along the wall surface with $\overline{\nu_1} = 0$, yields an expression for pressure as a function of the Lambert W.

B. Second Case

If $Br \sim O(1)$ then from Eq. (7) it can be concluded that

$$O(1): \overline{u_{0s}} = \frac{2-\sigma_\nu}{\sigma_\nu}Kn\frac{\partial \overline{u}}{\partial \overline{y}}|_{\overline{y}=\frac{1}{2}} \quad (16)$$

$$\frac{\partial \overline{u_0}}{\partial \overline{y}}|_{\overline{y}=0} = 0 \quad (17)$$

The integral constants c_1 and c_2 of the Eq. (11) can be determined from boundary conditions Eq. (16) and Eq. (17). Therefore,

$$\overline{u}_o(\overline{x}, \overline{y}) = \frac{-\epsilon Re}{8\gamma M^2}\frac{\partial \overline{p}_o}{\partial \overline{x}}\{1 - 4\overline{y}^2 + 4(\frac{2-\sigma_\nu}{\sigma_\nu})Kn\} \quad (18)$$

Solving Eq. (5) for $\overline{\nu_1}$ and considering, $\overline{\nu_1} = 0$ on the wall surface as the boundary condition, $\overline{\nu_1}$ can be expressed as

$$\overline{\nu_1} = \frac{\epsilon Re}{8\gamma M^2}\frac{1}{\overline{P}_o}[\frac{1}{2}(\frac{\partial^2 \overline{P}_o^2}{\partial \overline{x}^2})(\overline{y} - \frac{4}{3}\overline{y}^3) + 4Kn_o(\frac{2-\sigma_\nu}{\sigma_\nu})\frac{\partial^2 \overline{P}_o^2}{\partial \overline{x}^2}\overline{y}] \quad (19)$$

Evaluating the preceding equation along the wall surface with $\overline{\nu_1} = 0$ yields an expression for pressure along the side walls

$$\overline{P}_o(\overline{x}) = -6\frac{2-\sigma_\nu}{\sigma_\nu}Kn_o+$$

$$[(6\frac{2-\sigma_\nu}{\sigma_\nu}Kn_o)^2 + (1 + 12\frac{2-\sigma_\nu}{\sigma_\nu}Kn_o)\overline{x}$$

$$+(\overline{P_{in}}^2 + 12\frac{2-\sigma_\nu}{\sigma_\nu}Kn_o\overline{P_{in}})(1-\overline{x})]^{0.5} \quad (20)$$

V. RESULTS AND DISCUSSION

Velocity and pressure profiles are obtained for $Br \sim O(1)$ and $Br \sim O(\epsilon)$ for both wall heating and wall cooling cases. To obtain a useful solution from this study, the non-dimensionalized parameters are calculated for air with the following properties: $T = 350k, \gamma = 1.4, \mu = 208.2 * 10^{-7}\frac{N.s}{m^2}, \nu = 20.92 * 10^{-6}\frac{m^2}{s}, Pr = 0.7, u_{out} = 0.2\frac{m}{s}, \frac{P_{in}}{P_{out}} = 1.005, H = 1\mu m, L = 100\mu m, \sigma_\nu = 1$. As a result of the influence of fluid and channel properties, the fundamental parameters of the study are defined as, Re=0.01, M=0.001, Kn=0.08, and $\epsilon = 0.01$. For the first case, Br is equivalent to 0.01 and for the second Br would be 1 when q_s'' is 1.7 and 170 $\frac{w}{m^2}$, respectively. As it was shown by the equations the creeping effect becomes insignificant when $Br \sim O(1)$ however its effect becomes more pronounced when $Br \sim O(\epsilon)$.
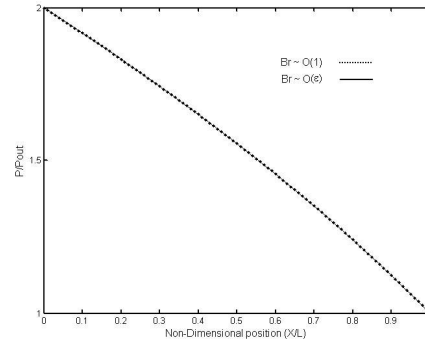

Fig. 2. Non-dimensionalized pressure vs. non-dimensionalized position for two different magnitudes of Br numbers for the heating wall condition.
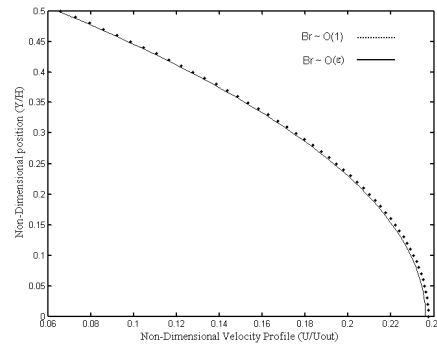

Fig. 3. Non-dimensionalized position vs. non-dimensionalized velocity for two different magnitudes of Br numbers for the heating wall condition.

Figures (2) and (3) reveals the creeping effect on the pressure and velocity profiles for heating wall condition. As Fig. (2) represents, the creeping effect on the pressure profile is negligible in the channel-wise direction. The non-dimensionalized velocity of fluid is graphed versus the non-dimensionalized channel height in order to inspect the creeping effect on velocity profile, see Fig. (3). As it can be seen for

heating wall boundary condition, the creeping effect increases both the velocity of the fluid and the flow rate, also. The creeping effect on pressure and velocity profiles for cooling wall condition are illustrated in Figs. (4) and (5). Figure (4) justifies the neglect of the creeping effect on the pressure profile. As Fig. (5) depicts, the creeping phenomenon decreases the velocity and the flow rate, both. Therefore, the flow rate is increased or decreased based on the heating or cooling condition of the wall, respectively. It must be highly noted that the magnitude of the heat flux has an influence on the quasi-isothermal assumption [15].
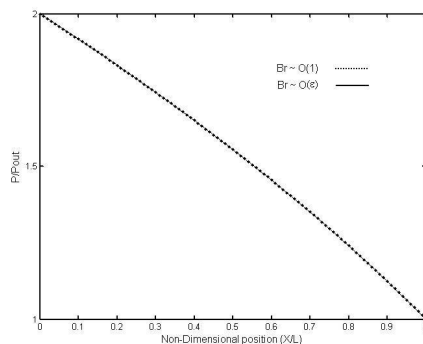

Fig. 4. Non-dimensionalized pressure vs. non-dimensionalized position for two different magnitudes of Br numbers for the cooling wall condition.
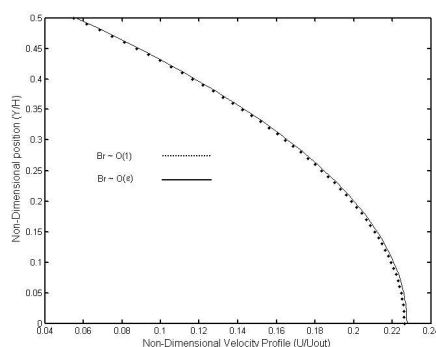

Fig. 5. Non-dimensionalized position vs. non-dimensionalized velocity for two different magnitudes of Br numbers for the heating wall condition.

VI. CONCLUSION

The goal of this work was to derive the analytical solution for the velocity profile of the compressible flow with constant heat flux boundary condition considering the creeping effect. The perturbation theory was applied to obtain the first order axial velocity, normal velocity, and pressure profiles. The creeping term in the slip equation was expressed based on the non-dimensional parameters including Pr, Re, M, and Br numbers. Simulations show that for studied cases there is not any difference between pressure profiles for the two Br numbers with different orders. However, some differences are observed between velocity profiles in two cases. Simulations show that for the heating wall condition the creeping term increases the velocity of the fluid, whereas for the cooling wall condition the creeping term has an inverse effect on the magnitude of the velocity.

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