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A high order Theory for Functionally Graded Shell

V. V. Zozulya

***Abstract*—**New theory for functionally graded (FG) shell based on expansion of the equations of elasticity for functionally graded materials (GFMs) into Legendre polynomials series has been developed. Stress and strain tensors, vectors of displacements, traction and body forces have been expanded into Legendre polynomials series in a thickness coordinate. In the same way functions that describe functionally graded relations has been also expanded. Thereby all equations of elasticity including Hook's law have been transformed to corresponding equations for Fourier coefficients. Then system of differential equations in term of displacements and boundary conditions for Fourier coefficients has been obtained. Cases of the first and second approximations have been considered in more details. For obtained boundary-value problems solution finite element (FE) has been used of Numerical calculations have been done with Comsol Multiphysics and Matlab**.**

***Keywords*—S**hell, FEM, FGM, Legendre polynomial

INTRODUCTION

RECENT years the FGMs have been applied in a science and engineering, as reflected in numerous papers [10, 11]. They are advantageous over classical homogeneous materials with only one material constituent, because FGMs consist of more material constituents and they combine the desirable properties of each constituent. As a representative example for FGMs, we just mention the metal/ceramic FGMs, which are compositionally graded from a ceramic phase to a metal phase. Metal/ceramic FGMs can incorporate advantageous properties of both ceramics and metals such as the excellent heat, wear, and corrosion resistances of ceramics and the high strength, high toughness, good machinability and bonding capability of metals without severe internal thermal stresses.

The FG thin-walled structures have numerous applications, especially in reactor vessels, turbines and many other applications in aerospace engineering [9]. Laminated composite materials are commonly used in many kinds of engineering structures. In conventional laminated composite structures, homogeneous elastic laminas are bonded together to obtain enhanced mechanical properties. However, the abrupt change in material properties across the interface between different materials can result in large interlaminar stresses leading to delamination [7]. One way to overcome these adverse effects is to use FGMs in which material properties vary continuously by gradually changing the volume fraction of the constituent materials. This eliminates interface problems of composite materials and thus the stress distributions are smooth.

V. V. Zozulya is with the *Centro de Investigacion Cientifica de Yucatan, A.C*, Merida, Yucatan, Mexico (phone: 999-942-8330; fax: 999-981-3900; e-mail: zozulya@ cicy.mx).

Various theories of FG plates and shells have been developed last decades [1, 2, 4, 6, 13]. The material properties of FG plates and shells can be described by various functional relations. Most researchers use the power-law function, exponential function, or sigmoid function [1, 2, 6] to describe the volume fractions. Models of FG plates and shells are based on the Kirchhoff-Love, Timoshenko-Mindlin hypothesis or used more complicated high order theories. Mathematically rigorous and promising for engineering applications approach to creation high order hierarchical models of plates and shells is based on expansion of the 3-D equations of elasticity in Legendre polynomials series in term of thickness. Such an approach have been used widely for development various theories of isotropic [3, 12] and anisotropic [5] plates and shells. The method of Legendre polynomials series expansion has been used widely in our previous publications for development theory of thermoelasticity of plates and shells with considering close mechanical and thermal contact [14-25]. More specifically, problem of heat conducting and unilateral contact of plates and shell through the heat-conducting layer with considering a change of layer thickness in the process of the shell deformation has been formulated in [14-16, 20, 24, 25]. The developed approach have been applied to the laminated composite materials with possibility of delamination and thermoelastic contact in temperature field in [17, 18,], the pencil-thin nuclear fuel rods modeling in [19] and some other engineering problems in [21-23].

In this paper we are developing new theory for FG shells based on expansion of the equations of elasticity for GFMs into Legendre polynomials series. More specifically, we expanded functions that describe functionally graded relations into Legendre polynomials series and find Hook's law that related Fourier coefficients for expansions of stress and strain Numerical examples are presented.

I. 3-D FORMULATION

Let a linear elastic body occupy an open in 3-D Euclidian space simply connected bounded domain $V \in \mathbf{R}^3$ with a smooth boundary ∂V. We assume that elastic body is inhomogeneous isotropic shell of arbitrary geometry with $2h$ thickness. The domain is $V = \Omega \times [-h, h]$ and it is embedded in in Euclidean space. Boundary of the shell can be presented in the form $\partial V = S \cup \Omega^+ \cup \Omega^-$. Here Ω is the middle surface of the shell, $\partial\Omega$ is its boundary, Ω^+ and Ω^- are the outer sides and $S = \Omega \times [-h, h]$ is a sheer side.

Stress-strain state of the elastic body is defined by stress σ^{ij} and ε_{ij} strain tensors and displacements u_i, traction p_i, and body forces b_i vectors. These quantities are not independent, they are related by equations of elasticity.

For convenience we transform above equations of elasticity taking into account that the radius vector $\mathbf{R}(\mathbf{x})$ of any point in domain *V*, occupied by material points of shell may be presented as

$$\mathbf{R}(\mathbf{x}) = \mathbf{r}(\mathbf{x}_\alpha) + x_3 \mathbf{n}(\mathbf{x}_\alpha) \tag{1}$$

where $\mathbf{r}(\mathbf{x}_\alpha)$ is the radius vector of the points located on the middle surface of shell, $\mathbf{n}(\mathbf{x}_\alpha)$ is a unit vector normal to the middle surface.

Let us consider that $\mathbf{x}_\alpha = (x^1, x^2)$ are curvilinear coordinates associated with main curvatures of the middle surface of the shell. In order to simplify 3-D equations of elasticity we introduce orthogonal system of coordinates related to main curvatures of the middle surface of the shell. Such coordinates are widely used in the shell theory. In this case the equations of equilibrium have the form

$$\begin{aligned}
&\frac{\partial(A_2\sigma_{11})}{\partial x_1} + \frac{\partial(A_1\sigma_{12})}{\partial x_2} + A_1A_2\frac{\partial\sigma_{13}}{\partial x_3} + \sigma_{12}\frac{\partial A_1}{\partial x_2} + \\
&+\sigma_{13}A_1A_2k_1 - \sigma_{22}\frac{\partial A_2}{\partial x_1} + A_1A_2b_1 = 0, \\
&\frac{\partial(A_2\sigma_{21})}{\partial x_1} + \frac{\partial(A_1\sigma_{22})}{\partial x_2} + A_1A_2\frac{\partial\sigma_{23}}{\partial x_3} + \sigma_{21}\frac{\partial A_2}{\partial x_1} + \\
&+\sigma_{23}A_1A_2k_2 - \sigma_{11}\frac{\partial A_1}{\partial x_2} + A_1A_2b_2 = 0, \\
&\frac{\partial(A_2\sigma_{31})}{\partial x_1} + \frac{\partial(A_1\sigma_{32})}{\partial x_2} + \frac{\partial(A_1A_2\sigma_{33})}{\partial x_3} - \\
&-\sigma_{11}A_1A_2k_1 - \sigma_{22}A_1A_2k_2 + A_1A_2b_3 = 0
\end{aligned} \tag{2}$$

Cauchy relations have the form

$$\begin{aligned}
&\varepsilon_{11} = \frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial x_2}u_2 + k_1u_3, \\
&\varepsilon_{22} = \frac{1}{A_2}\frac{\partial u_2}{\partial x_2} + \frac{1}{A_2A_1}\frac{\partial A_2}{\partial x_1}u_1 + k_2u_3, \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}, \\
&\varepsilon_{12} = \frac{1}{A_2}\left(\frac{\partial u_1}{\partial x_2} - \frac{1}{A_1}\frac{\partial A_2}{\partial x_1}u_2\right) + \frac{1}{A_1}\left(\frac{\partial u_2}{\partial x_1} - \frac{1}{A_2}\frac{\partial A_1}{\partial x_2}u_1\right) \\
&\varepsilon_{13} = \frac{\partial u_1}{\partial x_3} - k_1u_1 + \frac{1}{A_1}\frac{\partial u_3}{\partial x_1}, \varepsilon_{23} = \frac{\partial u_2}{\partial x_3} - k_2u_2 + \frac{1}{A_2}\frac{\partial u_3}{\partial x_2}
\end{aligned} \tag{3}$$

Here $A_\alpha(x_1, x_2) = \sqrt{\mathbf{r}(x_1, x_2)\cdot\mathbf{r}(x_1, x_2)}$ are coefficients of the first quadratic form of the middle surface of the shell, $k_\alpha(x_1, x_2)$ are it main curvatures.

In the case if inhomogeneous of the shell consists of graduation of the elastic modulus in the x_3 direction generalized Hook's law for FG elastic shell we represent in the form

$$\sigma_{ij}(\mathbf{x}) = c_{ijkl}(\mathbf{x})\varepsilon_{kl}(\mathbf{x}), \quad c_{ijkl}(\boldsymbol{x}) = E(\boldsymbol{x})c^0_{ijkl} \tag{4}$$

where for isotropic shell

$$c^0_{ijkl} = \lambda^0\delta_{ij}\delta_{kl} + 2\mu^0\delta_{il}\delta_{jk}, \mu^0 = \frac{1}{2(1+\nu)}, \lambda^0 = \frac{2\nu\mu^0}{(1-2\nu)} \tag{5}$$

Substituting Cauchy relations (3) in Hook's law (4) and then Hook's law into equations of equilibrium (2) we obtain differential equations of equilibrium in the form of displacements

$$A_{ij}(\boldsymbol{x})u_j(\boldsymbol{x}) + b_i(\boldsymbol{x}) = 0 \tag{6}$$

Here

$$A_{ij}(\boldsymbol{x}) = E(\boldsymbol{x})c^0_{ijkl}\partial_k\partial_l = E(\boldsymbol{x})A^0_{ij} \tag{7}$$

where A^0_{ij} is a differential operator that correspond to the case of homogeneous equations of elasticity.

These equations will be used for elaboration of the 2-D equations for FG shells.

II. 2-D FORMULATION

If Let us expand the parameters, that describe stress-strain of the cylindrical shell in the Legendre polynomials series along the coordinate x_3.

$$\begin{aligned}
&u_i(\boldsymbol{x}) = \sum_{k=0}^{\infty} u_i^k(\boldsymbol{x}_\alpha)P_k(\omega), \; u_i^k(\boldsymbol{x}_\alpha) = \frac{2k+1}{2h}\int_{-h}^{h} u_i(\boldsymbol{x}_\alpha, x_3)P_k(\omega)dx_3, \\
&\sigma_{ij}(\boldsymbol{x}) = \sum_{k=0}^{\infty} \sigma_{ij}^k(\boldsymbol{x}_\alpha)P_k(\omega), \; \sigma_{ij}^k(\boldsymbol{x}_\alpha) = \frac{2k+1}{2h}\int_{-h}^{h} \sigma_{ij}(\boldsymbol{x}_\alpha, x_3)P_k(\omega)dx_3, \\
&\varepsilon_{ij}(\boldsymbol{x}) = \sum_{k=0}^{\infty} \varepsilon_{ij}^k(\boldsymbol{x}_\alpha)P_k(\omega), \; \varepsilon_{ij}^k(\boldsymbol{x}_\alpha) = \frac{2k+1}{2h}\int_{-h}^{h} \varepsilon_{ij}(\boldsymbol{x}_\alpha, x_3)P_k(\omega)dx_3, \\
&p_i(\boldsymbol{x}) = \sum_{k=0}^{\infty} p_i^k(\boldsymbol{x}_\alpha)P_k(\omega), \; p_i^k(\boldsymbol{x}_\alpha) = \frac{2k+1}{2h}\int_{-h}^{h} p_i(\boldsymbol{x}_\alpha, x_3)P_k(\omega)dx_3, \\
&b_i(\boldsymbol{x}) = \sum_{k=0}^{\infty} b_i^k(\boldsymbol{x}_\alpha)P_k(\omega), \; b_i^k(\boldsymbol{x}_\alpha) = \frac{2k+1}{2h}\int_{-h}^{h} b_i(\boldsymbol{x}_\alpha, x_3)P_k(\omega)dx_3.
\end{aligned} \tag{8}$$

Substituting these expansions in equations (2)-(3) we obtain corresponding relations for Legendre polynomials series coefficients.

Equations of equilibrium have the form

$$\begin{aligned}
&\frac{\partial(A_2\sigma_{11}^k)}{\partial x_1} + \frac{\partial(A_1\sigma_{12}^k)}{\partial x_2} + \sigma_{12}^k\frac{\partial A_1}{\partial x_2} + \sigma_{13}^kA_1A_2k_1 - \\
&-\sigma_{22}^k\frac{\partial A_2}{\partial x_1} - \underline{\sigma_{13}^k} + A_1A_2f_1^k = 0, \\
&\frac{\partial(A_2\sigma_{21}^k)}{\partial x_1} + \frac{\partial(A_1\sigma_{22}^k)}{\partial x_2} + \sigma_{12}^k\frac{\partial A_2}{\partial x_1} + \sigma_{23}^kA_1A_2k_2 - \\
&-\sigma_{11}^k\frac{\partial A_2}{\partial x_1} - \underline{\sigma_{23}^k} + A_1A_2f_2^k = 0,
\end{aligned} \tag{9}$$

$$\frac{\partial\left(A_2\sigma_{31}^k\right)}{\partial x_1}+\frac{\partial\left(A_1\sigma_{32}^k\right)}{\partial x_2}-\sigma_{11}^k A_1 A_2 k_1-\sigma_{22}^k A_1 A_2 k_2-$$
$$-\underline{\sigma_{33}^k}+A_1 A_2 f_3^k=0 .$$

where

$$\underline{\sigma_{i3}^k}\left(\boldsymbol{x}_\alpha\right)=A_1 A_2\frac{2k+1}{h}\left(\sigma_{i3}^{k-1}\left(\boldsymbol{x}_\alpha\right)+\sigma_{i3}^{k-3}\left(\boldsymbol{x}_\alpha\right)+\ldots\right),$$
$$f_i^k\left(\boldsymbol{x}_\alpha\right)=b_i^k\left(\boldsymbol{x}_\alpha\right)+\frac{2k+1}{h}\left(\sigma_{i3}^{+}\left(\boldsymbol{x}_\alpha\right)-(-1)^k\sigma_{i3}^{-}\left(\boldsymbol{x}_\alpha\right)\right) \tag{10}$$

Cauchy relations have the form

$$\varepsilon_{11}^k=\frac{1}{A_1}\frac{\partial u_1^k}{\partial x_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial x_2}u_2^k+k_1u_3^k ,$$
$$\varepsilon_{22}^k=\frac{1}{A_2}\frac{\partial u_2^k}{\partial x_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial x_1}u_1^k+k_2u_3^k ,$$
$$\varepsilon_{12}^k=\frac{1}{A_2}\left(\frac{\partial u_1^k}{\partial x_2}-\frac{1}{A_1}\frac{\partial A_2}{\partial x_1}u_2^k\right)+\frac{1}{A_1}\left(\frac{\partial u_2^k}{\partial x_1}-\frac{1}{A_2}\frac{\partial A_1}{\partial x_2}u_1^k\right),$$
$$\varepsilon_{13}^k=\frac{1}{A_1}\frac{\partial u_3^k}{\partial x_1}-k_1u_1^k+\underline{u_1^k},$$
$$\varepsilon_{23}^k=\frac{1}{A_2}\frac{\partial u_3^k}{\partial x_2}-k_2u_2^k+\underline{u_2^k},\varepsilon_{33}=\underline{u_3^k}. \tag{11}$$

where

$$\underline{u_i^k}\left(\boldsymbol{x}_\alpha\right)=\frac{2k+1}{h}\left(u_i^{k+1}\left(\boldsymbol{x}_\alpha\right)+u_i^{k+3}\left(\boldsymbol{x}_\alpha\right)+\ldots\right) \tag{12}$$

In order to transform Hook's law in 1-D form we expand Young's $E(\mathbf{x})$ in Legendre polynomials series

$$E\left(\boldsymbol{x}\right)=\sum_{r=1}^{\infty}E^r\left(\boldsymbol{x}_\alpha\right)P_r(x_3),$$
$$E^k\left(\boldsymbol{x}_\alpha\right)=\frac{2k+1}{2h}\int_{-h}^{h}E\left(\boldsymbol{x}_\alpha,x_3\right)P_k(\omega)dx_3 . \tag{13}$$

Substituting this expansion and expansions for stress and strain tensors in Hook's law we obtain 1-D Hook's law for Legendre polynomials series coefficients

$$\sigma_{ij}^n\left(\boldsymbol{x}_\alpha\right)=c_{ijkl}^0\sum_{r=1}^{\infty}\sum_{m}^{\infty}\in^{nrm}E^r\left(\boldsymbol{x}_\alpha\right)\varepsilon_{kl}^m\left(\boldsymbol{x}_\alpha\right) \tag{14}$$

where

$$\in^{nrm}=\int_{-1}^{1}P_n(x_3)P_r(x_3)P_m(x_3)dx_3 \tag{15}$$

Substituting Cauchy relations (11) and Hook's law (14) in equations of equilibrium (9) we obtain differential equations in displacements. This system of equations contains infinite number of equations which are 2-D, they can be written in the form

$$\mathbf{E}\cdot\left(\mathbf{L}\cdot\mathbf{u}\right)=\mathbf{f} \tag{16}$$

$$\mathbf{E}=\begin{vmatrix}E_{ij}^{00} & E_{ij}^{01} & \cdots\\ E_{ij}^{10} & E_{ij}^{11} & \cdots\\ \vdots & \vdots & \ddots\end{vmatrix},\mathbf{L}=\begin{vmatrix}L_{ij}^{00} & L_{ij}^{01} & \cdots\\ L_{ij}^{10} & L_{ij}^{11} & \cdots\\ \vdots & \vdots & \ddots\end{vmatrix},\mathbf{u}=\begin{vmatrix}u_j^0\\ u_j^1\\ \vdots\end{vmatrix},\mathbf{f}=\begin{vmatrix}f_j^0\\ f_j^1\\ \vdots\end{vmatrix} \tag{17}$$

Here L_{ij}^{nm} are differential operators that correspond to homogeneous elastic shells, $E^{nm}=\in^{nrm}E^r$ are coefficients that characterized inhomogeneous properties of the shell.

Now instead of one 3-D system of the differential equations in displacements (6) we have of 2-D infinite differential equations for coefficients of the Legendre's polynomial series expansion. In order to simplify the problem approximate theory has to be developed and only finite set of members have to be taken into account in the expansion (8). Order of the system of equations depends on assumption regarding thickness distribution of the stress-strain parameters of the shell.

III. Results and discussion

We consider here the case of relatively thick shells. Therefore we will keep three members in polynomial expansion (8). In this case we will get the second order approximation equations for functionally graded shells. In this case the stress-strain parameters, which describe the state of the shell, can be presented in the form

$$\sigma_{ij}\left(\boldsymbol{x}\right)=\sigma_{ij}^0\left(\boldsymbol{x}_\alpha\right)P_0\left(\omega\right)+\sigma_{ij}^1\left(\boldsymbol{x}_\alpha\right)P_1\left(\omega\right)+\sigma_{ij}^2\left(\boldsymbol{x}_\alpha\right)P_2\left(\omega\right)$$
$$\varepsilon_{ij}\left(\boldsymbol{x}\right)=\varepsilon_{ij}^0\left(\boldsymbol{x}_\alpha\right)P_0\left(\omega\right)+\varepsilon_{ij}^1\left(\boldsymbol{x}_\alpha\right)P_1\left(\omega\right)+\varepsilon_{ij}^2\left(\boldsymbol{x}_\alpha\right)P_2\left(\omega\right),$$
$$u_i\left(\boldsymbol{x}\right)=u_i^0\left(\boldsymbol{x}_\alpha\right)P_0\left(\omega\right)+u_i^1\left(\boldsymbol{x}_\alpha\right)P_1\left(\omega\right)+u_i^2\left(\boldsymbol{x}_\alpha\right)P_2\left(\omega\right), \tag{18}$$
$$p_i\left(\boldsymbol{x}\right)=p_i^0\left(\boldsymbol{x}_\alpha\right)P_0\left(\omega\right)+p_i^1\left(\boldsymbol{x}_\alpha\right)P_1\left(\omega\right)+p_i^2\left(\boldsymbol{x}_\alpha\right)P_2\left(\omega\right),$$
$$b_i\left(\boldsymbol{x}\right)=b_i^0\left(\boldsymbol{x}_\alpha\right)P_0\left(\omega\right)+b_i^1\left(\boldsymbol{x}_\alpha\right)P_1\left(\omega\right)+b_i^2\left(\boldsymbol{x}_\alpha\right)P_2\left(\omega\right).$$

Taking into account formulae (15) for the coefficients $\in^{nrm}$ Hook's law for coefficients of the Legendre polynomials series expansion (15) has the form

$$\sigma_{ij}^0=c_{ijkl}^0\left(2E^0\varepsilon_{kl}^0+\frac{2}{3}E^1\varepsilon_{kl}^1+\frac{2}{5}E^2\varepsilon_{kl}^2\right),$$
$$\sigma_{ij}^1=c_{ijkl}^0\left(\frac{2}{3}E^1\varepsilon_{kl}^0+\left(\frac{2}{3}E^0+\frac{4}{15}E^2\right)\varepsilon_{kl}^1+\frac{4}{15}E^1\varepsilon_{kl}^2\right),$$
$$\sigma_{ij}^2=c_{ijkl}^0\left(\frac{2}{5}E^2\varepsilon_{kl}^0+\frac{4}{15}E^1\varepsilon_{kl}^1+\left(\frac{2}{5}E^0+\frac{4}{35}E^2\right)\varepsilon_{kl}^2\right). \tag{19}$$

Now system of equations for displacements has the same form as (16), but it contains only four equations and corresponding matrixes and vector have the form

$$E=\begin{vmatrix}E_{ij}^{00} & E_{ij}^{01} & E_{ij}^{02}\\ E_{ij}^{10} & E_{ij}^{11} & E_{ij}^{12}\\ E_{ij}^{02} & E_{ij}^{12} & E_{ij}^{22}\end{vmatrix} \tag{20}$$

$$E_{ij}^{00}=\begin{vmatrix}2E^0 & 0\\ 0 & 2E^0\end{vmatrix}, E_{ij}^{01}=\begin{vmatrix}\frac{2}{3}E^1 & 0\\ 0 & \frac{2}{3}E^1\end{vmatrix}, E_{ij}^{02}=\begin{vmatrix}\frac{2}{5}E^2 & 0\\ 0 & \frac{2}{5}E^2\end{vmatrix},$$

$$E_{ij}^{01}=\begin{vmatrix}\frac{2}{3}E^1 & 0\\ 0 & \frac{2}{3}E^1\end{vmatrix}, E_{ij}^{11}=\begin{vmatrix}\frac{2}{3}E^0+\frac{4}{15}E^2 & 0\\ 0 & \frac{2}{3}E^0+\frac{4}{15}E^2\end{vmatrix},$$

$$E_{ij}^{12}=\begin{vmatrix}\frac{4}{15}E^1 & 0\\ 0 & \frac{4}{15}E^1\end{vmatrix}, E_{ij}^{20}=\begin{vmatrix}\frac{2}{3}E^2 & 0\\ 0 & \frac{2}{3}E^2\end{vmatrix}, E_{ij}^{21}=\begin{vmatrix}\frac{4}{15}E^1 & 0\\ 0 & \frac{4}{15}E^1\end{vmatrix}$$

$$E_{ij}^{22}=\begin{vmatrix}\frac{2}{5}E^0+\frac{4}{35}E^2 & 0\\ 0 & \frac{2}{5}E^0+\frac{4}{35}E^2\end{vmatrix}$$

$$\mathbf{L}=\begin{vmatrix}L_{11}^{00} & L_{13}^{00} & 0 & L_{13}^{01} & 0 & 0\\ L_{31}^{00} & L_{33}^{00} & L_{31}^{01} & L_{33}^{01} & 0 & 0\\ L_{11}^{10} & 0 & L_{11}^{11} & L_{13}^{11} & 0 & L_{13}^{12}\\ L_{31}^{10} & L_{33}^{10} & L_{31}^{11} & L_{33}^{11} & 0 & L_{33}^{12}\\ 0 & 0 & 0 & L_{13}^{21} & L_{11}^{22} & L_{13}^{22}\\ 0 & 0 & L_{31}^{21} & L_{33}^{21} & L_{31}^{22} & L_{33}^{22}\end{vmatrix}, \mathbf{u}=\begin{vmatrix}u_1^0\\ u_3^0\\ u_1^1\\ u_3^1\\ u_1^2\\ u_3^2\end{vmatrix}, \mathbf{f}=\begin{vmatrix}f_1^0\\ f_3^0\\ f_1^1\\ f_3^1\\ f_1^2\\ f_3^2\end{vmatrix} \quad (21)$$

Most of operators L_{ij}^{nm} are differential, they have the form

$$L_{11}^{00}u_1^0=(\lambda+2\mu)\frac{\partial^2 u_1^0}{\partial x_1^2}, L_{13}^{00}u_3^0=\frac{\lambda}{R}\frac{\partial u_3^0}{\partial x_1}, L_{11}^{01}u_1^1=0,$$

$$L_{13}^{01}u_3^1=\frac{\lambda}{h}\frac{\partial u_3^1}{\partial x_1}, L_{11}^{02}u_1^2=0, L_{12}^{02}u_2^2=0, L_{31}^{00}u_1^0=-\frac{\lambda}{R}\frac{\partial u_1^0}{\partial x_1},$$

$$L_{33}^{00}u_3^0=\mu\frac{\partial^2 u_2^0}{\partial x_1^2}-(\lambda+2\mu)\frac{u_3^0}{R}, L_{33}^{02}u_3^2=0, L_{11}^{10}u_1^0=0,$$

$$L_{13}^{10}u_3^0=-\frac{3\mu}{h}\frac{\partial u_2^0}{\partial x_1}, L_{11}^{11}u_1^1=(\lambda+2\mu)\frac{\partial^2 u_1^1}{\partial x_1^2}-\frac{3\mu}{h^2}u_1^1,$$

$$L_{13}^{11}u_3^1=\frac{\lambda}{R}\frac{\partial u_3^1}{\partial x_1}+\mu k_1\frac{\partial u_2^1}{\partial x_1}, L_{31}^{00}u_1^0=-\frac{\lambda}{R}\frac{\partial u_1^0}{\partial x_1}, L_{33}^{02}u_3^2=0,$$

$$L_{33}^{00}u_3^0=\mu\frac{\partial^2 u_2^0}{\partial x_1^2}-(\lambda+2\mu)\frac{u_3^0}{R}, L_{11}^{10}u_1^0=0,$$

$$L_{13}^{10}u_3^0=-\frac{3\mu}{h}\frac{\partial u_2^0}{\partial x_1}, L_{11}^{11}u_1^1=(\lambda+2\mu)\frac{\partial^2 u_1^1}{\partial x_1^2}-\frac{3\mu}{h^2}u_1^1,$$

$$L_{13}^{11}u_3^1=\frac{\lambda}{R}\frac{\partial u_3^1}{\partial x_1}+\mu k_1\frac{\partial u_2^1}{\partial x_1}, L_{11}^{12}u_1^2=0, L_{13}^{12}u_3^2=\frac{3\lambda}{h}\frac{\partial u_3^2}{\partial x_1},$$

$$L_{31}^{10}u_1^0=-\frac{3\lambda}{h}\frac{\partial u_1^0}{\partial x_1}, L_{33}^{10}u_3^0=-\frac{3\lambda}{Rh}u_3^0, L_{31}^{11}u_1^1=-\frac{\lambda}{R}\frac{\partial u_1^1}{\partial x_1},$$

$$L_{33}^{11}u_3^1=\mu\frac{\partial^2 u_2^1}{\partial x_1^2}-(\lambda+2\mu)\left(\frac{1}{R^2}+\frac{3}{h^2}\right)u_3^1, L_{31}^{12}u_1^2=0,$$

$$L_{33}^{12}u_3^2=-\frac{3\lambda}{Rh}u_3^2, L_{11}^{20}u_1^0=0, L_{13}^{20}u_3^0=0, L_{11}^{21}u_1^1=0,$$

$$L_{33}^{21}u_3^1=-\frac{5\mu}{h}\frac{\partial u_3^1}{\partial x_1}, L_{11}^{22}u_1^2=(\lambda+2\mu)\frac{\partial^2 u_1^2}{\partial x_1^2}-\frac{15\mu}{h^2}u_1^2,$$

$$L_{13}^{22}u_3^2=\frac{\lambda}{R}\frac{\partial u_3^2}{\partial x_1}, L_{21}^{20}u_1^0=0, L_{33}^{20}u_3^0=0, L_{12}^{21}u_1^1=-\frac{5\lambda}{h}\frac{\partial u_1^1}{\partial x_1}, \quad (22)$$

$$L_{33}^{21}u_3^1=-\frac{5\lambda}{Rh^2}u_3^1, L_{21}^{22}u_1^2=-\frac{\lambda}{R}\frac{\partial u_1^2}{\partial x_1},$$

$$L_{22}^{22}u_2^2=\mu\frac{\partial^2 u_2^2}{\partial x_1^2}-(\lambda+2\mu)\left(\frac{1}{R^2}+\frac{15}{h^2}\right)u_3^2$$

Substituting these operators into (21) we obtain system of differential equations which together with corresponding boundary conditions can be used for the stress-strain calculation for the second approximation shell theory.

Material properties of an FGM are the functions of volume fractions and they are managed by a volume fraction. When the shell is considered to consist of two materials with Young's modulus E_1 and E_2 respectively, the effective Young's modulus $E(x_3)$ given by the following power-law expression

$$E(x_3)=(E_2-E_1)\left(\frac{x_3+h}{2h}\right)^n+E_1 \quad (n\geq 0) \qquad (23)$$

Substituting function (23) into equation (13) we obain expressions for the Legendre polynomials coefficients for the effective Young's modulus

$$E^1=\frac{(E_2+E_1)n}{1+n}, E^2=\frac{(E_2-E_1)nh}{2+3n+n^2},$$
$$E^3=-\frac{5(E_2-E_1)(n-1)nh^2}{(1+n)(2+n)(3+n)} \qquad (24)$$

For simplisity in this study we consider dimetionless coordinates $\xi_1=\frac{x_1}{L}$ and $\xi_3=\frac{x_3}{h}$ have been introdused. Calculations have been done for Young's modulus equal to $E_1=1\,\mathrm{Pa}$ and $E_1/E_2=2$ and for Poison ratio $\nu=0.3$ respectively, other paramiters are $R=0.25L$, $h=0.25R$ and $n=0.2$. Numerical calculations have been done using commercial software Comsol Multiphysics and Matlab. Results of calculations are presented on Fig. 1 – Fig. 3.

Fig. 1 shows the Legendre polynomials coefficients for the displacements distribution versus the normalized length for the second approximation theory. These coefficients are FEM solutions of the systems of differential equations (16) with differential operators (22). Fig. 2 shows displacements and stresses distribution versus normalized length and thickness for second approximation theory.

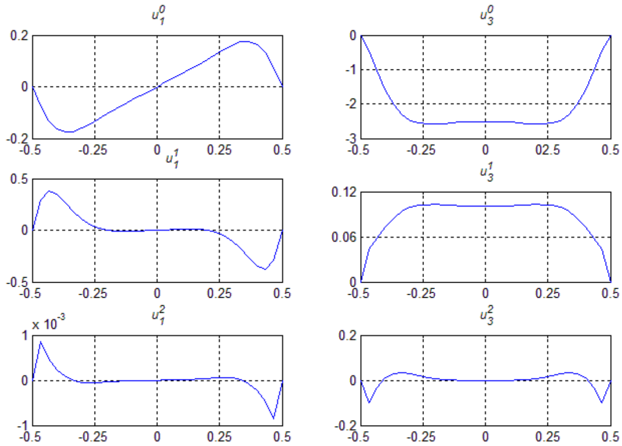

Fig.1 mLegendre polynomials coefficients for the displacements

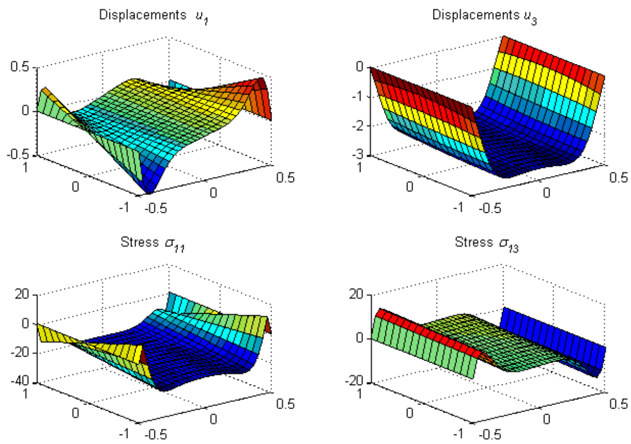

Fig. 2 Displacements and stresses versus normalized length and thickness

On Fig. 3 are presented values of maximal displacements u_3 and stresses σ_{33} at the cross section situated at the middle point of the shell with coordinate $\xi_1 = 0$ versus ratio h/R.

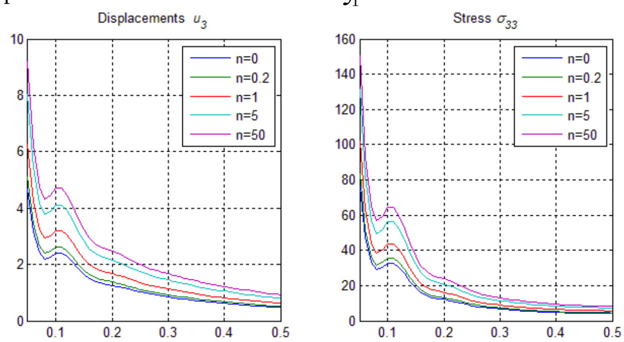

Fig. 3 Displacements u_3 and stresses σ_{33} versus h/R for various exponents n

IV. Conclusion

The high order theory for FG axisymmetric cylindrical shell based on expansion of the axisymmetric equations of elasticity for FMs into Legendre's polynomials series has been developed. Starting from axisymmetric equations of elasticity for FGMs, The stress and strain tensors, vectors of displacements, traction and body forces and also function that describe functionally graded relations for Young's modulus have been expanded into Legendre polynomials series in term of the shell thickness coordinate. Then all equations of elasticity including Hook's law have been transformed to corresponding equations for the Legendre's polynomials series expansion coefficients. The system of differential equations in term of displacements and boundary conditions for the coefficients of expansion has been obtained. Cases of the first and second approximations have been considered in more details. All necessary equations and heir coefficients has been written explicitly and corresponding boundary-value problems have been formulated. For numerical solution of the formulated problems finite element (FE) has been used and commercial software Comsol Multiphysics and Matlab have been used. For validation of the proposed theory and obtained equations comparison with results obtained using equations of elasticity has been done for exponential function for graduation law. Influence of different parameters on the stress-strain state of the cylindrical shell has been studied.

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