

# Improvement of the Shortest Path Problem with Geodesic-Like Method

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**Abstract**—This paper proposes a method to improve the shortest path problem on a NURBS (Non-uniform rational basis spline) surfaces. It comes from an application of the theory in classic differential geometry on surfaces and can improve the distance problem not only on surfaces but in the Euclidean 3-space  $\mathbb{R}^3$ .

**Keywords**—shortest paths, geodesic-like method, NURBS surfaces.

## I. INTRODUCTION

A fundamental problem in many fields such as CAD, CAGD, robotics and computer graphics etc. is to ask how to compute effectively the distance between two objects on a surface. This problem can be present as simple as to find the minimum distance between two geometric objects  $\mathbb{R}^3$ , but it is hard to improve in general and in practice.

The simplest case about this problem is to compute the distance between two points and it can be estimated exactly by the Pythagorean theorem. The orthogonal projection problem considers the case that only one object is a point and has many applications[20], [19]. There are many results in the investigations of the orthogonal projection problem. Especially, in order to improve the distance problem between a point and a NURBS curve, Chen[3], [4], MaYL [16] and Selimovic[26] presented some effective methods. Hu[10] developed a good method to improve the orthogonal projection onto curves and surfaces. In the case that none of these objects is a single point, Kim[13] presented a method to estimate the distance between a canal surface and a simple surface in 2003, while Chen[5] improved this problem on two implicit algebraic surfaces in 2006. Therefore, one can find many methods to investigate the shortest path problem in  $\mathbb{R}^3$ .

However, there are only few methods to study this problem on a curved surface. Maekawa [17] shows a very good method for solving the shortest path and the orthogonal projection problems on free-form parametric surfaces. Generally, the distance problem on a regular surface is more complicated than that in  $\mathbb{R}^3$ , even though the distance is just between two points on the surface. It is equivalent to find the length of the shortest path between them. We can find more information in reference [18]. This classical problem has many applications, such as in object segmentation, multi-scale image analysis and CAD etc.[2], [14], [25]. There are also many methods to estimate the shortest path on triangular mesh [15], [27], polyhedral[11], [22] and regular surface[12], [24] etc. These methods can be extend to improve the distance between one

point and one curve or between two curves on surface but they are not effective methods.

In this paper, the distance problem on NURBS surfaces and parametric surfaces will be improved by the geodesic-like method with B-spline basis shown in [6] and [7]. In fact, the geodesic-like method can also improve the distance problem in  $\mathbb{R}^3$  but its efficiency is less than other algorithms that we known. In section 2, we will review the related geodesic theory in the classic differential geometry. The notion of geodesic-like method will be introduced in section 3. In section 4 we shall present how to use the geodesic-like algorithm to estimate the distance between two objects on a NURBS surface. The discussion of the shortest path problem and some simulations will be presented in the last of this paper.

## II. GEODESICS IN CLASSIC DIFFERENTIAL GEOMETRY

A geodesic on a regular surface  $S$  is a smooth curve  $\alpha(s) : [a, b] \rightarrow S$  satisfying that the covariant derivative of  $\alpha$  vanishes at each point. That is,  $\gamma$  satisfies the differential equation

$$\frac{D}{ds}\gamma' = 0, \quad (1)$$

where  $\frac{D}{ds}$  is the covariant derivative and  $\gamma'$  is the derivative of  $\gamma$  with respect to the parameter  $s$ . Precisely, Suppose that  $S$  is a regular surface and  $(U, \mathbf{x})$  is a system of coordinates on  $S$ . A curve  $\gamma(t) = (x_1(t), x_2(t))$  is a geodesic curve in  $(U, \mathbf{x})$  on  $S$  if it satisfies the system of geodesic equations (see [1])

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt} \frac{dx_j}{dt} = 0, \quad k = 1, 2. \quad (2)$$

Again, we consider a regular surface  $S$  with a parametrization  $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow S$ . Here it is known that  $\mathbf{x}$  is a diffeomorphism from  $U$  onto a subset  $\mathbf{x}(U)$  of  $S$ . Then the energy function on a smooth curve  $\gamma$  on the surface  $S$  with a parameter  $\gamma(s) : [a, b] \rightarrow U$  is defined by

$$E(\gamma) = \frac{1}{2} \int_a^b \left\| \frac{d}{ds} \mathbf{x}(\gamma(s)) \right\|^2 ds. \quad (3)$$

A proper variation of the curve  $\gamma$  is a differentiable map  $h : [a, b] \times [-\varepsilon, \varepsilon] \rightarrow U$  such that

$$\begin{cases} h(s, 0) = \gamma(s), & s \in [a, b] \\ h(a, t) = \gamma(a), & t \in [-\varepsilon, \varepsilon] \\ h(b, t) = \gamma(b), & t \in [-\varepsilon, \varepsilon] \end{cases} \quad (4)$$

Intuitively,  $h_t(\cdot) = h(\cdot, t)$  is a differentiable curve with the same endpoints for each  $t \in (-\varepsilon, \varepsilon)$ . Thus the energy function on  $h_t$  can be represented as

$$E(t) = \frac{1}{2} \int_a^b \left\| \frac{\partial(\mathbf{x} \circ h)}{\partial s}(s, t) \right\|^2 ds, \quad t \in (-\varepsilon, \varepsilon). \quad (5)$$

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From the theory of differential geometry, it is well-known that the critical points of the function  $E(t)$  is the geodesics on surfaces i.e. if  $E'(0) = 0$  then  $\gamma$  is a geodesic on  $S$ .

Here is a basic relationship between the length and energy functions.

**Theorem 1:** Let  $S$  be a regular surface and  $p, q \in S$  be two distinct points. If  $\gamma$  is a shortest path between  $p$  and  $q$  on  $S$ , then  $\gamma$  is a geodesic on  $S$  which pass through  $p$  and  $q$ . That is, the geodesic  $\gamma(t)$  is a critical point of the length function

$$L(s) = \int_a^b \left\| \frac{\partial f}{\partial t}(s, t) \right\| dt. \quad (6)$$

The distance between two points on a surface  $S$  is defined by the length of minimum path on  $S$  from  $p$  to  $q$ . Then

$$d(p, q) = \min_{\gamma \in \Gamma} L(\gamma). \quad (7)$$

where  $\Gamma$  is the set of all paths on  $S$  from  $p$  to  $q$  and  $L(\gamma)$  is the length of the curve  $\gamma$  on  $S$ . From theorem 1, the set  $\Gamma$  can be only considered the set of all geodesic on  $S$  from  $p$  to  $q$ . See [1] for details.

### III. GEODESICS-LIKE METHOD

Consider the parametric surface  $S$  with a parametrization  $\mathbf{x} : U \rightarrow \mathbb{R}^3$  and two curves  $c_1, c_2$  on  $S$ . For simplicity, we still denote  $c_1, c_2 : [a, b] \rightarrow U$  such that  $\mathbf{x}(c_1([a, b]))$ ,  $\mathbf{x}(c_2([a, b]))$  are the curves  $c_1$  and  $c_2$ , respectively. Thus the distance between  $c_1$  and  $c_2$  on  $S$  can be computed by

$$d(c_1, c_2) = \min_{s, t \in [a, b]} d(\mathbf{x}(c_1(s)), \mathbf{x}(c_2(t))). \quad (8)$$

That is,  $d(c_1, c_2)$  is the length of minimum geodesic from  $c_1$  to  $c_2$ . Equation (8) introduces a simple algorithm to improve this distance problem but it is too expansive. Let us describe it roughly.

**Algorithm 1:** First, we digitize the curves  $c_1$  and  $c_2$  to two sequences of points,  $\{p_i\}_{i=0}^m$  and  $\{q_j\}_{j=0}^n$ , respectively. For each  $i, j$ , estimating the minimum geodesic  $\gamma_{ij}$  between  $p_i$  and  $q_j$ . Then the shortest path in  $\{\gamma_{ij}\}_{(i,j)=(0,0)}^{(m,n)}$  approaches the minimum geodesic between  $c_1$  and  $c_2$  on surface  $S$  when  $m, n$  are large enough. Of course its length approaches the minimum distance between  $c_1$  and  $c_2$  on  $S$ .

Solve the geodesic between two fixed points is crucial to solve Algorithm 1. One can find many effective methods in the references [9], [12], [15], [22], [24], [27]. However, if the numbers of  $\{p_i\}$  and  $\{q_j\}$  are large, this algorithm becomes very slow. In fact, algorithm 1 is the simplest and the slowest method to improve this problem.

**Definition 2:** Let  $\mathbf{x}(u, v)$  be a parametrization of a regular surface  $S$ ,  $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow S$ . A curve  $\tilde{c}(s)$  on  $U$  is called a geodesic-like curve of order  $n + 1$  on  $S$  if  $\tilde{c}(s) = \sum_{i=0}^n N_i^n(s)(\tilde{u}_i, \tilde{v}_i)$  is a B-spline curve and satisfies the system of geodesic equations

$$(\nabla E)(\tilde{u}_i, \tilde{v}_i) = 0, \quad (9)$$

where

$$E(u_i, v_j) = \frac{1}{2} \int_a^b \|\mathbf{x}(c(t))\|^2 dt$$

is the energy function of curve

$$c(t) = \sum_{i=0}^n N_i^n(t)(u_i, v_i)$$

and  $(\nabla E)(u_i, v_j)$  is the gradient of  $E(u_i, v_j)$ .

Equation (9) is called the system of standard geodesic-like equations. Although the system of geodesic-like equations are integral equations, they can be improved by the Newton's method, the iterator method or other numerical methods[8], [12], [21], [23], [28] effectively.

Since any piecewise differential curve can be approximated by the B-spline curves, a geodesic-like curve approaches a geodesic on  $S$  when the order of geodesic-like curve is large enough. In the other words, we can estimate the distance between two points on  $S$  via the minimum geodesic-like curves. We summarize this property as follows.

**Theorem 3:** Let  $S$  be a parametric surface and let  $\gamma : [0, 1] \rightarrow S$  be a geodesic. Assume that the curve  $c_n = \sum_{i=0}^n N_i^n(t)(u_i, v_i)$  is the geodesic-like curve between  $\gamma(0)$  and  $\gamma(1)$  for each positive integer  $n \geq 2$ . Then

$$\lim_{n \rightarrow \infty} c_n = \gamma. \quad (10)$$

The system of geodesic-like equations provides a elegant method to improve the distance problem between two objects on surfaces. We are now in a position to introduce this method in this section. The parametrization  $\mathbf{x}$  on  $S$  is defined on  $U = [a, b] \times [c, d]$ . That is

$$\mathbf{x} : [a, b] \times [c, d] \rightarrow S \subset \mathbb{R}^3.$$

Let  $c_1$  and  $c_2$  be two differentiable parameterized curves on  $S$  and

$$\begin{aligned} c_1(s) &: [0, 1] \rightarrow [a, b] \times [c, d] \\ c_2(t) &: [0, 1] \rightarrow [a, b] \times [c, d]. \end{aligned}$$

Thus  $c_1 = \mathbf{x}(c_1([0, 1]))$  and  $c_2 = \mathbf{x}(c_2([0, 1]))$  are two curves on  $S$ . To exclude the zero distance case from our consideration, we can assume that the two curves have no intersection. Denote  $c_1(s) = (u_0(s), v_0(s))$  and  $c_2(t) = (u_n(t), v_n(t))$  where  $u_0, u_n : [0, 1] \rightarrow [a, b]$  and  $v_0, v_n : [0, 1] \rightarrow [c, d]$  are all differentiable functions. Note that a B-spline curve from  $[0, 1]$  into  $[a, b] \times [c, d]$  with  $c(0) \in c_1$  and  $c(1) \in c_2$  always has the form as

$$\begin{aligned} c(x) &= \sum_{i=1}^{n-1} N_i^n(x)(u_i, v_i) + N_0^n(x)c_1(s) + N_n^n(x)c_2(t) \\ &= \sum_{i=1}^{n-1} N_i^n(x)(u_i, v_i) + N_0^n(x)(u_0(s), v_0(s)) \\ &\quad + N_n^n(x)(u_n(t), v_n(t)) \end{aligned} \quad (11)$$

where  $x \in [0, 1]$ .

Hence, we can rewrite the system of geodesic-like equations to the following three different forms. These formulas improve the distance between two curves on  $S$ , the orthogonal projection problem on  $S$  and the shortest path

between two points on  $S$ , respectively.

*The system of geodesic-like equations between two curves:*  
From the equation (11), the parameters of the energy function  $E$  are  $s, t, u_1, \dots, u_{n-1}, v_1, \dots, v_{n-1}$ . The system of geodesic-like equations between two curves can be rewritten as

$$\begin{aligned} (\nabla E) \\ = (E_s, E_t, E_{u_1}, E_{u_2}, \dots, E_{u_{n-1}}, E_{v_1}, E_{v_2}, \dots, E_{v_{n-1}}) \\ = 0 \end{aligned} \quad (12)$$

*The system of geodesic-like equations between one point and one curve:*

If  $c_1$  is a constant curve on  $S$ , then the derivative of  $E$  about  $t$  is vanish. Thus we obtain the geodesic-like equation between one point and one curve.

$$\begin{aligned} (\nabla E) \\ = (E_t, E_{u_1}, E_{u_2}, \dots, E_{u_{n-1}}, E_{v_1}, E_{v_2}, \dots, E_{v_{n-1}}) \\ = 0 \end{aligned} \quad (13)$$

Of course, The orthogonal projection problem on surface can be improved by equation (13).

*The system of geodesic-like equations between two points:*  
Moreover, if  $c_1$  and  $c_2$  are both constant curves on  $S$ , then the geodesic-like equations between points is

$$\begin{aligned} (\nabla E) \\ = (E_{u_1}, E_{u_2}, \dots, E_{u_{n-1}}, E_{v_1}, E_{v_2}, \dots, E_{v_{n-1}}) \\ = 0 \end{aligned} \quad (14)$$

A curve satisfies one of equations (12) - (14) is called a geodesic-like curve between  $c_1$  and  $c_2$ . Let us describe how to find the local minimum geodesic-like curve between two curves  $c_1$  and  $c_2$  on the surface  $S$ . In this algorithm, we solve the system of geodesic-like curve equations by the Newton's method and the iterator method.

*Algorithm 2: (Geodesic-like algorithm)*

**Step 1:** Given two closed curves  $c_1$  and  $c_2$  on the surface. Input an initial curve  $c$  such that the endpoints of  $c$  are on the  $c_1$  and  $c_2$ .

**Step 2:** Solving the geodesic-like equations (equation (12) or (14)) by the initial curve  $c$  and obtain a geodesic-like curve, which we still denote it by  $c$ , between  $c_1$  and  $c_2$ .

**Step 3:** If the set  $(c \cap c_1) \cup (c \cap c_2)$  consists of the endpoints of  $c$ , then  $c$  is the local minimum geodesic-like curve between  $c_1$  and  $c_2$ . Otherwise, trimming away some parts of the curve  $c$  such that the intersections of this trimmed curve, which we still denote it by  $c$ , and  $c_1 \cup c_2$  are only the endpoints of this trimmed curve. Then repeat step 2.

By Theorem (3), one will proceed by the geodesic-like algorithm to obtain the shortest path between  $c_1$  and  $c_2$  when  $n$  is large enough. We summarize it as follows.

*Theorem 4:* Let  $S$  be a parametric surface and  $c_1, c_2$  be two closed curves on  $S$ . For each  $n \geq 2$ ,  $\tilde{c}_n$  is the local

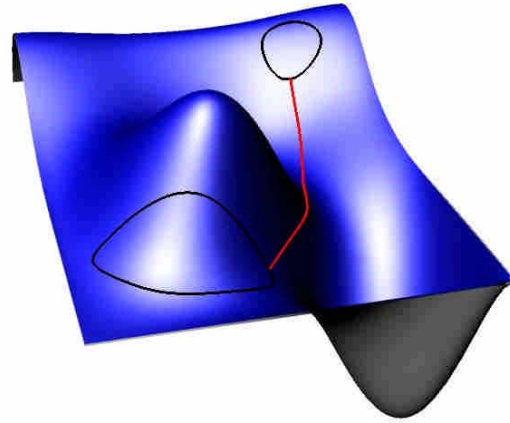


Fig. 1 Distance between two closed curves on a NURBS surface

minimum geodesic-like curve that obtained by the geodesic-like algorithm (algorithm 2). If the set  $\{\tilde{c}_n\}$  is a convergent sequence, then there exists a local minimum geodesic  $\gamma$  between  $c_1$  and  $c_2$  such that

$$\lim_{n \rightarrow \infty} \tilde{c}_n = \gamma. \quad (15)$$

Moreover,  $\tilde{c}_n$  is orthogonal to  $c_1$  and  $c_2$  when  $n$  is large enough.

#### IV. EXAMPLES AND DISCUSSION

The following two examples show the applications of the geodesic-like method in practice. In our simulations, the geodesic-like curves are all uniform quadratic B-spline curves in  $\mathbb{R}^2$ .

First we consider an open surface  $S$  and two closed curves  $c_1$  and  $c_2$  on  $S$  as in Figure 1. The surface  $S$  is a cubic B-spline surface with (8,4) control points. The red curve in figure 4 is the local minimum geodesic-like curve of order 11 between  $c_0$  and  $c_1$  and its error is less than  $10^{-6}$ .

Secondly, we construct a face model as in Figure 2 by NURBS surface and find the minimum geodesic-like curves between two holes in the model. The red curve in figure 2 is the local minimum geodesic-like curve of order 30 and the green curve is the exact minimum geodesic between two holes. In Figure 3, we propose that the geodesic-like algorithm has increased actually this simulation efficiency. The order in Figure 3 means the number of control points. Then the lengths of geodesic-like curves constructed by our method approaches the minimum distance between the two holes. Especially, the percentage of error, which

$$\text{error}(\%) = \frac{\text{Length} - \text{minimum distance}}{\text{minimum distance}} \times 100\%, \quad (16)$$

will be less than  $10^{-7}$  provided the geodesic-like curve is constructed by 60 control points.

The geodesic-like algorithm provides an effective and reliable computation of shortest paths between two curves on surfaces. For computing the shortest paths between two curves

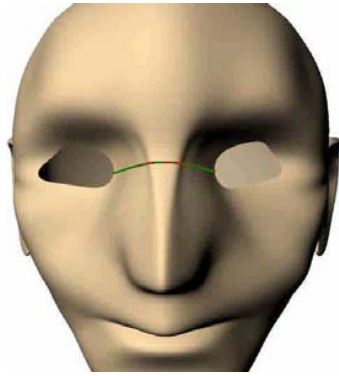


Fig. 2 The distance between two holds on a face model

minimum distance :	16.5411205				
Order	3	4	5	6	7
Length	17.5720035	17.4468728	17.0309897	16.9571156	16.8484675
Time	0.1009	0.14	0.266	0.422	0.484
error(%)	6.23224406	5.47576145	2.96152368	2.51491488	1.85807848
Order	9	10	11	12	13
Length	16.7433831	16.6780621	16.6571845	16.6094243	16.573226
Time	1.2	3.03	3.4	5.2	3.828
error(%)	1.22278657	0.82788587	0.70166952	0.41293333	0.19409507
Order	15	16	17	20	30
Length	16.5487338	16.5543877	16.5495022	16.5439818	16.5413563
Time	7.213	7.192	7.228	9	15
error(%)	0.04602651	0.08020738	0.0506719	0.0172981	0.00142554

Fig. 3 The table of the distance between two holds on a face model with different orders

on  $\mathbb{R}^3$ , our method is comparable with other well-known methods. Especially, the construction of geodesic-like curves only bases on the uniform quadratic B-spline curves since it is enough to us to consider the geodesic-like curves in the plane. Significantly, our method can be extended to solve the distance problem between any two objects on surfaces and the distance problem in higher dimension.

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