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Sparse Frequencies Extracting from Partial Phase-Only Measurements

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Abstract—This paper considers a robust recovery of sparse frequencies from partial phase-only measurements. With the proposed method, sparse frequencies can be reconstructed, which makes full use of the sparse distribution in the Fourier representation of the complex-valued time signal. Simulation experiments illustrate the proposed method's advantages over conventional methods in both noiseless and additive white Gaussian noise cases.

Keywords—Sparse signal recovery, phase-only measurements, Compressive sensing, convex relaxation.

I. Introduction

IN general, the frequency components of a complex signal generally cannot be recovered from just the magnitude or phase alone. But it can be reconstructed from partial knowledge of the input signal under certain conditions. For instance, when the signal is band limited, it can be recovered using extrapolation from the limited number of known samples. Besides, under the constraint of minimum phase, the magnitudes and phases of frequency components are related by the Hilbert transform. They are no longer independent with each other. Signal reconstruction using phase-only measurements has been studied extensively [1]-[5],[10].

In [1]-[2], two algorithms are proposed to reconstruct a real discrete-time signal within a scale factor from the phase of its Fourier transform. However, those methods are not suitable for complex-value signal. Moreover, the issue of extracting signal frequencies from phase-only data has also been investigated in [3]. But all of the methods are restricted in the number of frequencies. As the extension of previous works (i.e., [1]-[3]), Lee et al. have addressed the problem of extracting multiple frequencies within a scale factor from phase-only data of a complex discrete-time signal [4]. However, given partial phase-only data, these methods of [4] are still invalid. Recently, Liu et al. have investigated sparse support recovery with phase-only measurements by exploiting compressive sensing (CS) technique in [5] (The details of CS can refer to [6]-[9]). As it only exploits the phase components of the measurements in the constraint, it can avoid the performance deterioration by corrupted amplitude components. However, they don't use the amplitude information and consider the problem of spectrum estimation. Different from previous literature, we address the problem of extracting multiple frequencies from compressive phase-only data in [10]. The frequency components of the complex signal reconstructed from compressive phase-only data are obtained by solving an optimization problem, but it is only suitable for noiseless case. Although an iterative hard threshold algorithm is proposed for noisy case, it cannot obtain acceptable performance with single snapshot.

The contributions of the paper can be summarized as follows: Different from [4] to solve an overdetermined system, sparse frequencies can be extracted from partial phase-only data. It is recovered by solving an underdetermined system with sparsity constraint. Besides, also different from our previous work in [10], the proposed optimization method can also work in noisy cases with single snapshot.

The following notations are used throughout the paper. Matrices and vectors are represented by bold uppercase and bold lowercase characters, respectively. Vectors are by default in column orientation, whereas Re$\{\cdot\}$ and Im$\{\cdot\}$ return real and imaginary parts of the input, respectively. $\mathbf{X}_{kn}$ is (k,l)-th element of the matrix $\mathbf{X}$. $|\cdot|$ and $||\cdot||_p$ stand for magnitude operator and l_p-norm respectively. R_+ is the set $(0,\infty)$. $||\mathbf{X}||_F$ is the Frobenius norm of the matrix $\mathbf{X}$. $diag\{\cdot\}$ is diagonalization operator produces a diagonal matrix from a column vector. Symbol $<\mathbf{a},\mathbf{b}>$ means inner production between $\mathbf{a}$ and $\mathbf{b}$ (i.e., $\mathbf{a}^H\mathbf{b}$). $\mathbf{X}(l,:)$ is l-th row of the matrix $\mathbf{X}$.

II. Problem Formulation

From the M-point discrete Fourier transform (DFT), the finite-length signal $x(n)$ satisfies the following equation

$$x(n)=\frac{1}{M}\sum_{k=0}^{M-1}X(\omega_k)e^{j2\pi nk/M}, n=0,1,\cdots,M-1 \quad (1)$$

where $\omega_k = 2\pi k/M$. $X(\omega_k)$ is the DFT of $x(n)$, $k = 0,1,\cdots,M-1$. Meanwhile, written in polar form, $x(n)$ is represented in terms of its magnitude and phase as

$$x(n)=|x(n)|\,e^{j\phi_x(n)} \quad (2)$$

where $|x(n)|$ and $\phi_x(n)$ are the magnitude and phase of $x(n)$, respectively. From equations (1) and (2), we have

$$|x(n)|\,e^{j\phi_x(n)}=\frac{1}{M}\sum_{k=0}^{M-1}X(\omega_k)e^{j2\pi nk/M} \quad (3)$$

And thus

$$|x(n)|=\frac{1}{M}\sum_{k=0}^{M-1}X(\omega_k)e^{j(2\pi nk/M-\phi_x(n))} \quad (4)$$

Up to here, equation (4) is the problem formulation we interested in the paper. Our problem is how to reconstruct $\mathbf{X}(\omega_k)$ only using the phase data $\phi_x(n)$.

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III. Problem Solution

For the above equation (4), one possible approach for recovery from the phase-only measurements involves the solution of a set of linear equations. The frequency estimation problem can be transformed to solve a set of linear equations from the phase samples $\phi_x(n)$. Frequencies of signal can be recovered from phase-only data by solving [4]:

$$\min_{\mathbf{x}} || \,\mathrm{Im}\{\mathbf{Px}\}||_2$$
$$s.t. \begin{cases} \mathrm{Re}\{\mathbf{x}(0)\} = \alpha, \ and \ \alpha > 0 \\ \mathbf{x}(k) = 0, \ for \ k = N, N+1, \cdots, M-1 \\ \mathbf{x} \in C^M \end{cases} \tag{5}$$

where $\mathbf{P}_{kn} = \frac{1}{M}\exp(j(2\pi kn/M - \phi_x(n)))$ and $k, n = 0, 1, \ldots, M-1$, $M \geqslant 2N-1$. $\mathbf{P} \in C^{M\times M}, \mathbf{x} = [X(\omega_0), X(\omega_1), \cdots, X(\omega_{M-1})]^T$. Frequency components can also be extracted from phase-only data with an iterative algorithm in [4]. We regard it as Lee's method.

If we just obtain $Q(Q < M)$ random phase-only measurements from ϕ_x(n) in (4), frequency components cannot be reconstructed with (5). However, noted that frequencies $\mathbf{x}$ are sparse, motivated by (4) and [10], a natural strategy for recovering $\mathbf{x}$ is to search object function with l_0-norm minimize subject to constraints as follows,

$$\min_{\mathbf{x}} ||\mathbf{x}||_0$$
$$s.t. \begin{cases} || \ \mathrm{Im}\{\mathbf{P}'\mathbf{x}\}||_2 \leqslant \sigma \\ \mathrm{Re}\{\mathbf{P}'\mathbf{x}\} \succeq \varepsilon\mathbf{e}, \ \mathbf{x} \in C^M \end{cases} \tag{6}$$

where $\mathbf{e} \in R_+^Q$, and matrix $\mathbf{P}' \in C^{Q\times M}$ is a matrix whose rows are picked randomly from matrix $\mathbf{P}$ in (5). For extracting multiple frequencies from partial phase-only data, the amplitude of $\mathbf{x}$ is vague. Therefore, it is insensitive to choose parameter ε. It just requires a positive scale (i.e., $\varepsilon \in R_+$). Similarly, it requires that the entries in $\mathbf{e}$ are positive because of the amplitude ambiguity of $\mathbf{x}$. Without loss of generality, we set $\mathbf{e} = [1, 1, \cdots, 1]^T$. σ is the maximum acceptable error and $\sigma \geqslant 0$.

However, it is an NP-hard problem to solve (6). In standard CS, the objective function l_0-norm minimization can be replaced by l_1-norm minimization if sensing matrix has restricted isometry property (RIP). As the standard constraints in CS literature are different from those in (6), RIP of sensing matrix with the constraints in (6) doesn't imply constrained l_0-norm minimization is equivalent to its corresponding constrained l_1-norm minimization, but standard CS relaxation strategy can give us an important insight. Similarly with the analysis of standard CS, we illustrate the matrix $\mathbf{P}'$ has RIP as follows.

Lemma 1: [11] Considering the $M \times M$ DFT matrix $\mathbf{W}$ with entries $\mathbf{W}_{k,n} = \frac{1}{\sqrt{M}}\exp\{-j2\pi kn/M\}$ $k, n \in \{0, 1, \cdots, M-1\}$, the matrix $\mathbf{W}'$ whose rows are independent picked from matrix $\mathbf{W}$ (Algebraically, we can view $\mathbf{W}'$ as a random row sub-matrix of the DFT matrix $\mathbf{W}$), if the number of rows Q satisfies $Q \geqslant c\delta^{-2}K\log(M)\log^3(K)\}$ with probability at least $1 - 2M\exp(-\frac{c\delta^2 Q}{2M})$ for all $||\mathbf{x}||_0 \leqslant K$, we have $(1-\delta)||\mathbf{x}||_2 \leqslant ||\mathbf{W}'\mathbf{x}||_2 \leqslant (1+\delta)||\mathbf{x}||_2$. Here $c > 0$ is an absolute constant which only depends on K, $0 < \delta < 1$.

Theorem 1: Assume $\mathbf{P} \in C^{M\times M}$ is DFT matrix, i.e., $\mathbf{P}_{kn} = \frac{1}{\sqrt{M}}\exp(j2\pi kn/M - j\phi_x(n))$, $k, n \in \{0, 1, \cdots, M-1\}$, and suppose matrix $\mathbf{P}'$ is random matrix whose rows are independent picked from matrix $\mathbf{P}$. If the number of rows Q satisfies $Q \geqslant c\delta^{-2}K\log(M)\log^3(K)\}$, for all $||\mathbf{x}||_0 \leqslant K$ with probability at least $1 - 2M\exp(-\frac{c\delta^2 Q}{2M})$, we have $(1-\delta)||\mathbf{x}||_2 \leqslant ||\mathbf{P}'\mathbf{x}|| \leqslant (1+\delta)||\mathbf{x}||_2$. Here $c > 0$ is an absolute constant which only depends on K, $0 < \delta < 1$.

Proof: We set $\mathbf{h} = \mathbf{P}'\mathbf{x}$ and $\mathbf{s} = \mathbf{W}'\mathbf{x}$, and denote $\mathbf{h} = [h_0, h_1, \cdots, h_{Q-1}]^T$ and $\mathbf{s} = [s_0, s_1, \cdots, s_{Q-1}]^T$, respectively. Without loss of generality, denoting 0-th component's index in $\mathbf{h}$ is produced by $< \mathbf{P}(\xi_0, :), \mathbf{x} >$, 1-th component's index in $\mathbf{h}$ is produced by $< \mathbf{P}(\xi_1, :), \mathbf{x} >$, $\cdots$,$(Q-1)$-th component's index in $\mathbf{h}$ is produced by $< \mathbf{P}(\xi_{Q-1}, :), \mathbf{x} >$. Similarly, denoting 0-th component's index in $\mathbf{s}$ is produced by $< \mathbf{W}(\xi_0, :), \mathbf{x} >$, 1-th component's index in $\mathbf{s}$ is produced by $< \mathbf{W}(\xi_1, :), \mathbf{x} >$, $\cdots$,$(Q-1)$-th component's index in $\mathbf{s}$ is produced by $< \mathbf{W}(\xi_{Q-1}, :), \mathbf{x} >$, respectively. For any l ($l = 1, 2, \cdots, Q-1$), we have

$$\begin{aligned} h_l &= |(\mathbf{P}'\mathbf{x})_l|^2 = |(\mathbf{P}\mathbf{x})_{\xi_l}|^2 = \\ &= \left|\sum_{t=0}^{M-1} \exp\left(j\phi_x(\xi_l) - \frac{2\pi\xi_l t}{M}\right) x_t\right|^2 \\ &= \left|\sum_{t=0}^{M-1} \exp\left(j\phi_x(\xi_l)\right) \exp\left(-\frac{2\pi\xi_l t}{M}\right) x_t\right|^2 \\ &= \left|\sum_{t=0}^{M-1} \exp\left(-\frac{2\pi\xi_l t}{M}\right) x_t\right|^2 \\ &= |(\mathbf{W}\mathbf{x})_{\xi_l}|^2 = |(\mathbf{W}'\mathbf{x})_l|^2 = s_l \end{aligned} \tag{7}$$

According to (7) and *Lemma 1*, the *Theorem 1* is proved. ■

Motivated by *Theorem 1* together with l_1-norm minimization relaxation in standard CS, we employ a relaxed version to recover sparse frequencies as follows.

$$\min_{\mathbf{x}} ||\mathbf{x}||_1$$
$$s.t. \begin{cases} || \ \mathrm{Im}\{\mathbf{P}'\mathbf{x}\}||_2 \leqslant \sigma \\ \mathrm{Re}\{\mathbf{P}'\mathbf{x}\} \succeq \varepsilon\mathbf{e}, \ \mathbf{x} \in C^M \end{cases} \tag{8}$$

where $\mathbf{e} \in R_+^Q$, $P' \in C^{Q\times M}$. Since (8) is a convex programming problem, the unique global minimum of the programming can be obtained. The simulated experiments in section IV will illustrate the validity of (8). For simplification, we regard (8) as phase-only frequency estimation algorithm (POFEA). In the following, we address the values of σ in both noiseless and additive white Gaussian noise (AWGN) cases. Obviously, $\sigma = 0$ for noiseless case. For noisy case, the received phase data in matrix $\mathbf{P}'$ is perturbed by noise. $\mathbf{D}'$ is the perturbation matrix and $\mathbf{D}' = diag\{[\exp(-j\phi'_x(0)), \exp(-j\phi'_x(1)), \cdots, \exp(-j\phi'_x(Q-1))]\}$. We have

$$\begin{aligned} &\mathrm{Im}\{\mathbf{D}'\mathbf{P}'\mathbf{x}\} = \mathrm{Im}\{\mathbf{D}'\{\mathrm{Re}\{\mathbf{P}'\mathbf{x}\} + j\mathrm{Im}\{\mathbf{P}'\mathbf{x}\}\}\} \\ &= \mathrm{Im}\{\{\mathrm{Re}\{\mathbf{D}'\} + j\,\mathrm{Im}\{\mathbf{D}'\}\}\,\mathrm{Re}\{\mathbf{P}'\mathbf{x}\}\} \\ &= \mathrm{Im}\{\mathbf{D}'\}\,\mathrm{Re}\{\mathbf{P}'\mathbf{x}\} = \mathrm{Im}\{\mathbf{D}'\}\mathbf{y} \end{aligned} \tag{9}$$

TABLE I: probability of successful recovery in noiseless case with different frequencies number

frequency number	number of measurements	Lee's method	POFEA
K=2	Q=43	98.4%	100%
	Q=60	99.6%	100%
	Q=77	100%	100%
K=4	Q=43	34%	100%
	Q=60	70%	100%
	Q=77	91%	100%
	Q=94	98%	100%
	Q=111	98.8%	100%
	Q=128	100%	100%

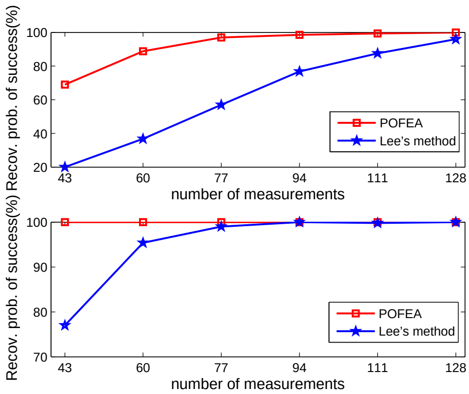

Fig. 1: PSR versus number of phase-only measurements (upper subplot SNR=5dB, bottom subplot SNR=10dB)

In (9), $\mathbf{P}' = \text{Re}\{\mathbf{P}'\} + j\text{Im}\{\mathbf{P}'\}$, $\mathbf{D}' = \text{Re}\{\mathbf{D}'\} + j\text{Im}\{\mathbf{D}'\}$, $\text{Re}\{\mathbf{P}'\mathbf{x}\} = \mathbf{y}$ and $\text{Im}\{\mathbf{P}'\mathbf{x}\} = \mathbf{0}$ are exploited, which yields

$$\begin{aligned} &||\,\text{Im}\{\mathbf{D}'\}\mathbf{y}||_2 \leqslant ||\,\text{Im}\{\mathbf{D}'\}||_F||\mathbf{y}||_F \\ &= \sqrt{\sum_{n=0}^{Q-1} \sin^2(\phi_x{}'(n))}||\mathbf{y}||_F \leqslant \sqrt{\sum_{n=0}^{Q-1} \phi_x{}'^2(n)}||\mathbf{y}||_F \\ &= \sqrt{\mu}||\mathbf{y}||_2/\sqrt{SNR} \end{aligned} \tag{10}$$

In (10), we define $\mu \triangleq \sum_{n=0}^{Q-1} \phi_x{}^2(n)$ and using signal-to-noise ratio definition $SNR \triangleq \sum_{n=0}^{Q-1} \phi_x{}^2(n)/\sum_{n=0}^{Q-1} \phi_x'{}^2(n)$. Furthermore, considering amplitude ambiguity of $\mathbf{y}$, without loss of generality, we assume $\mathbf{y}$ is a normalized vector (i.e., $||\mathbf{y}||_2 = 1$). Thus we set $\sigma = \sqrt{\mu/SNR}$. Besides, given $||\mathbf{y}||_2 = 1$ and $\mathbf{e} = [1, 1, \cdots, 1]^T$, parameter ε can be chosen from the interval $(0, Q^{-1/2}]$.

IV. Numerical Examples

In this section, we present two examples to illustrate the performance improvement of the proposed method. We compare the probability of success recovery (PSR) of POFEA with Lee's method. The one-dimensional complex sequence $x(n)$ of length $M = 128$ and $N = 64$. In POFEA, we take $\varepsilon = \frac{1}{2\sqrt{Q}}$. Suppose that spectrum of complex signals $x(n)$ are restricted in $\{\mathbf{x}(0), \mathbf{x}(1), \cdots, \mathbf{x}(N-1)\}$, whose frequencies are 0 for $k = N, N+1, \cdots, M-1$, and $\text{Re}\{\mathbf{x}(0)\} = 1$. Each magnitudes of complex exponentials in frequency domain are 100. In simulation, 500 independent Monte Carlo experiments are taken and all frequencies are assumed at the discrete grids.

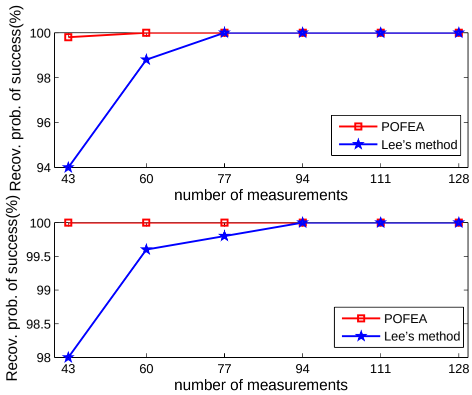

Fig. 2: PSR versus number of phase-only measurements (upper subplot SNR=15dB, bottom subplot SNR=20dB)

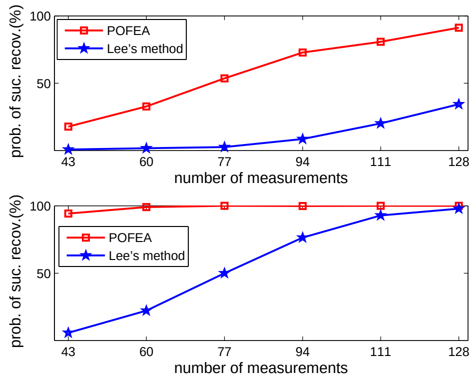

Fig. 3: PSR versus number of phase-only measurements (upper subplot SNR=5dB, bottom subplot SNR=10dB)

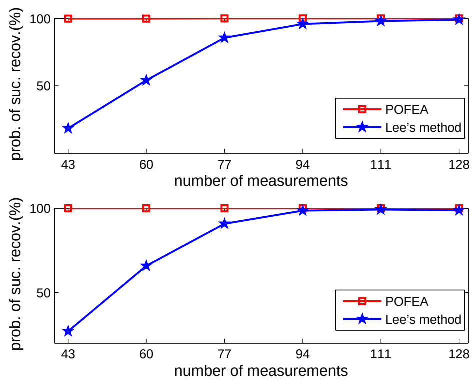

Fig. 4: PSR versus number of phase-only measurements (upper subplot SNR=15dB, bottom subplot SNR=20dB)

The PSR is defined as

$$p = \frac{D(\hat{x} = \text{x}_0)}{T} \tag{11}$$

where $\hat{x}$ is the estimated solution and x_0 is the generated true vector. Symbol $D(\Theta)$ means the number of times that the event

TABLE II: PSR in noisy case with different frequencies number

SNR	frequency number	POFEA	Lee's-full	Lee's-loss
5dB	1	99.6%	100%	93.2%
	2	69%	96%	20%
	3	37.6%	71.8%	2%
	4	17.8%	34.4%	0.8%
10dB	1	100%	100%	100%
	2	100%	100%	77%
	3	99.2%	100%	24.4%
	4	94.2%	97.8%	5.8%
15dB	1	100%	100%	100%
	2	99.8%	100%	94%
	3	100%	100%	51%
	4	100%	100%	18.4%
20dB	1	100%	100%	100%
	2	100%	100%	98%
	3	100%	100%	60.8%
	4	100%	100%	27%

Θ happens overall Mont Carlo trails, where T is the number of times of Mont Carlo trails.

Example 1: In this example, suppose frequency number in $x(n)$ is 2 (i.e., K=2). We evaluate the PSR of POFEA and Lee's method in noiseless case. Simulation results are shown in TABLE I. Meanwhile, We evaluate the PSR of POFEA and Lee's method at SNR=5dB,10dB,15dB and 20dB, respectively. Simulation results are show in Fig. 1 and Fig. 2, respectively. Besides, we also evaluate the PSR of two methods when frequency number in $x(n)$ equals 4 (i.e., K=4). Simulation results are shown in TABLE I for noiseless case and Fig. 3 and Fig. 4 for noisy cases, respectively.

TABLE I shows both POFEA and Lee's method can work in noiseless case when frequency number is 2, even if Q ($Q = 43$) phase-only samples can be used. Meanwhile, according to the results of Fig. 1 and Fig. 2, if we exploit partial phase-only data to extract frequencies, the PSR of POFEA is more robust to noise than Lee's method. Moreover, TABLE I also shows that POFEA can give the same PSR as Lee's method with full phase-only data when K=4. However, the performance of Lee's method deteriorates when number of phase-only data decreased. It illustrate that the proposed method is more robust to partial phase-only measurements than the Lee's method in [4]. In Fig. 3 and Fig. 4, we note that if we exploit partial phase-only data to recover frequencies, the performance of POFEA is better than Lee's method. When $SNR \geqslant 10dB$, the POFEA using partial phase-only data can work approximately the same as the Lee's method with full phase-only data. Besides, for same phase-only measurement number, Fig. 3 and Fig. 4 also indicate that POFEA is more robust to noise than Lee's method versus different values of SNR.

Example 2: In the example, considering $x(n)$ contains K (K=1,2,3,4) frequencies. Assume Q (Q=43) phase-only data are exploited. We simulate POFEA and Lee's method (denoted by Lee's-loss) with Q phase-only data at $SNR =$ 5dB, 10dB, 15dB and 20dB, respectively. For contrast, we also simulate the result of Lee's method with all M (M=128) phase-only data (denoted by Lee's-full) at $SNR =$5dB, 10dB, 15dB and 20dB, respectively. Simulation results are shown in TABLE II. The results of TABLE II indicate that if we exploit partial phase-only data to recover frequencies, the Lee's method cannot work when $K > 2$. While for SNR>10dB, the POFEA can work approximately the same as the Lee's method with full phase-only data.

Remark: Although we set $\mathrm{Re}\{X(\omega_0)\} \neq 0$ and $X(\omega_0) = 0$ for $k = N, N+1, \cdots, M-1$ in simulation, it's can recover multiple frequencies without these constraints in POFEA.

V. Conclusion

In the paper, we propose a robust sparse frequencies extraction method by reconstructing the spectrum of the complex-valued time signal with partial phase-only data. The simulation experiments show that the performance of POFEA outperforms conventional method.

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References

[1] M. Hayes, L. Jae and A. Oppenheim, "Signal reconstruction from phase or magnitude," *IEEE Trans. Acoust. Speech Signal Process.,* vol. 28, no.6 pp. 672-680, Dec. 1980.
[2] M. Hayes, "The reconstruction of a multidimensional sequence from the phase or magnitude of its Fourier transform," *IEEE Trans. Acoust. Speech Signal Process.,* vol. 30, no. 2, pp. 140-154, Apr. 1982.
[3] S. Kay, "A fast and accurate single frequency estimator," *IEEE Trans. Acoust. Speech Signal Process.,* vol. 37, no. 12 pp. 1987-1990, Dec. 1989.
[4] D.W. Lee, J.H. McClellan, "Extracting multiple frequencies from phase-only data," *IEEE Trans. Aerosp. Electron. Syst.,* vol. 33, no. 3, pp. 795-801, Jul. 1997.
[5] Y.P. Liu, Q. Wan, F. Wen, J. Xu, Y.N. Peng, "Sparse Support Recovery with Phase-Only Measurements," *arXiv: 1005.1801pdf*, 2010.
[6] D.L. Donoho, X. Huo, "Uncertainty principles and ideal atomic decomposition," *IEEE Trans. Inform. Theory,* vol.47, no.7, pp.2845-2862, Nov. 2001.
[7] E.J. Candes, J. Romberg, "Practical signal recovery from random projections," *Manuscript*, Jan., 2005.
[8] E.J. Candes, et al., "Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information," *IEEE Trans. Inform. Theory,* vol. 52, no.2, pp. 489-509, Feb. 2006.
[9] Y.C. Eldar, G. Kutyniok, "Compressed Sensing: Theory and Applications," *Cambridge University Press,* 2012.
[10] R. Fan, Q. Wan, H. Chen, et al, "Multiple frequencies estimation from compressive phase-only data: algorithm and application," *accepted to International Journal of Electronics.*, 2012. DOI: 10.1080/00207217.2012.743229.
[11] R. Vershynin, "Introduction to the non-asymptotic analysis of random matrices", *arXiv:1011.3027v7,* 2011.