

# The Dividend Payments for General Claim Size Distributions under Interest Rate

Li-Li Li, Jinghai Feng and Lixin Song

**Abstract**—This paper evaluates the dividend payments for general claim size distributions in the presence of a dividend barrier. The surplus of a company is modeled using the classical risk process perturbed by diffusion, and in addition, it is assumed to accrue interest at a constant rate. After presenting the integro-differential equation with initial conditions that dividend payments satisfies, the paper derives a useful expression of the dividend payments by employing the theory of Volterra equation. Furthermore, the optimal value of dividend barrier is found. Finally, numerical examples illustrate the optimality of optimal dividend barrier and the effects of parameters on dividend payments.

**Keywords**—Dividend payout, Integro-differential equation, Jump-diffusion model, Volterra equation

## I. INTRODUCTION

In this paper, we assume an insurance company faces two types of risk: one is a Poisson risk representing large movements due to claims and the other is a Brownian risk signifying the uncertainty to premium income or additional uncertainty to aggregate claims. We describe the risk process without dividends or investments as a classical risk process perturbed by diffusion

$$U_t = x + \mu t + \sigma W_t - \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0.$$

Here  $x \geq 0$  is the initial surplus,  $\mu > 0$  is the rate at which the premium is received and  $\sigma > 0$  is the volatility of the cash flow.  $\{N(t)\}_{t \geq 0}$  is a Poisson process with parameter  $\lambda > 0$ , denoting the total number of claims up to time  $t$ . The claim sizes  $Y_i$  independent of  $\{N(t)\}_{t \geq 0}$  and the Brownian motion  $\{W_t\}_{t \geq 0}$  are positive i.i.d. random variables with common distribution function  $P(x)$ , density function  $p(x)$ , and finite moment  $EY$ .

We now enrich the model. We allow the surplus to accrue interest at a constant rate  $\delta \geq 0$ , and assume that dividends are paid to the shareholders according to “barrier strategy”, i.e., there is a horizontal barrier of level  $b \geq 0$  such that when the surplus reaches level  $b$ , the “overflow” will be paid as dividends. Let  $X_b(t)$  be the modified surplus process with initial surplus  $X_b(0-) = x$  under the above barrier strategy,

and given by

$$dX_b(t) = (\mu + \delta X_{t-}) dt + \sigma dW_t - d \sum_{i=1}^{N(t)} Y_i - dL_t, \quad (1)$$

$$X_b(0-) = x,$$

where  $L_t$  is the cumulative amount of dividends paid by time  $t$ . Then the total expected present value of dividends with discount factor  $\rho \geq 0$  and initial surplus  $x$  equals

$$V_b(x) = E \left[ \int_0^{\tau_b} e^{-\rho t} dL_t \right],$$

where  $\tau_b := \inf \{t \geq 0, X_b(t) \leq 0\}$  is the bankruptcy time. To simplify notation we define  $V_b(x) = 0$  for  $x < 0$ . It is clear that  $V_b(x)$  is an increasing function of  $x$ .

Dividend payments provide an opportunity of profit participation for the shareholders of an insurance company. The barrier strategy considered in this paper was initially proposed by De Finetti [4] for binomial models and more general barrier strategies have been studied in many papers, e.g., [1], [3], [5], [6], [7], [9].

Paulsen and Gjessing [11] obtained the formulation of dividend payments using confluent hypergeometric function. They got the formula in the special case that interest rate is constant and surplus process is compound Poisson with exponentially distributed claims. Li [8] studied the problem in risk process (1) without interest rate. He got the explicit solution of dividend payments when claim sizes are exponentially distributed. Although these elegant results on formulation for special distributions have been obtained, few results for general cases have been given in the past studies.

It is necessary to discuss the dividend payments for general claim size distributions. Exponential distribution is only fit for describing a kind of claim event. There are other distributions for claim sizes in different insurance situations. For example, Gamma distribution has been used on occasion for automobile physical damage, and Pareto distribution is appropriate for the fire insurance case. Unfortunately, both methods of [8] and [11] do not apply to the case for general claim size distributions with constant interest rate.

In this paper, our main goal is to evaluate the dividend payments for general claim size distributions. We study the problem in the classical risk process perturbed by diffusion with constant interest rate. We first give initial conditions satisfied by the dividend payments. Then we derive its transparent formulation by employing the theory of Volterra equation. Based on this fact, we find the optimal value of dividend barrier. Numerical examples are given to illustrate the results

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and the effects of interest rate and diffusion volatility on dividend payments.

II. INTEGRO-DIFFERENTIAL EQUATION AND INITIAL CONDITIONS

In this section, we first give the integro-differential equation for  $V_b(x)$  using the result of [11] directly, and then we transform the boundary conditions to initial ones. The following theorem is a special case of Theorem 2.1 in [11].

*Theorem 2.1:* Let  $g(x)$  be bounded and twice continuously differentiable on  $(0, b)$  with bounded first derivative. If  $g(x)$  solves

$$\frac{1}{2}\sigma^2 g''(x) + (\mu + \delta x)g'(x) - (\lambda + \rho)g(x) + \lambda \int_0^x g(x-y)p(y)dy = 0, \quad 0 < x < b, \quad (2)$$

together with the conditions

$$g(0) = 0, \quad g'(b) = 1, \quad (3)$$

$$g(x) = g(b) + x - b \quad \text{for } x > b, \quad (4)$$

then  $g(x) = E[\int_0^{T_b} e^{-\rho t} dL_t]$ .

Theorem 2.1 provides a way to find the formulation of  $V_b(x)$ . But the integro-differential equation with boundary conditions given by (3) are difficult to solve. In fact, we can transform the problem to the one with initial conditions. Now we consider  $V'_b(0)$ , which is proved to be finite by the following lemma.

*Lemma 2.2:* The derivative of  $V_b(x)$  at  $x = 0$  is a positive number, that is, there exists a real number  $C \in (0, +\infty)$  such that  $V'_b(0) = C$ .

*Proof:* Since  $V_b(x)$  is an increasing function which implies  $V'_b(x) \geq 0$  for  $x \geq 0$ , we need only prove that  $V'_b(0) \neq 0$  and  $V'_b(0) \neq \infty$ . It is clear that  $V_b(x)$  satisfies conditions of Theorem 2.1.

Let  $M(r) = \int_0^\infty e^{-rx} V_b(x) dx$  and  $\widehat{P}(r) = \int_0^\infty e^{-rx} p(x) dx$ . Taking Laplace transforms on both sides of equation (2), we get

$$M'(r) + M(r) \left[ \frac{1}{r} - \frac{\sigma^2}{2\delta} r - \frac{\mu}{\delta} - \frac{\lambda \widehat{P}(r) - \rho - \lambda}{r\delta} \right] = -\frac{\sigma^2}{2r\delta} V'_b(0). \quad (5)$$

By introducing a new variable

$$a(r) = \int_1^r -\frac{\sigma^2}{2\delta} s - \frac{\mu}{\delta} - \frac{\lambda \widehat{P}(s) - \rho - \lambda}{s\delta} ds,$$

we can rewrite (5) as

$$rM'(r) + M(r) + M(r)ra'(r) = -\frac{\sigma^2}{2\delta} V'_b(0).$$

Multiplying each term of the above equation by  $e^{a(r)}$ , we have

$$\frac{d}{dr} (M(r)re^{a(r)}) = -\frac{\sigma^2}{2\delta} V'_b(0)e^{a(r)}.$$

Since  $\lim_{r \rightarrow \infty} M(r)re^{a(r)} = 0$ , it follows

$$M(r)re^{a(r)} = V'_b(0) \int_r^\infty \frac{\sigma^2}{2\delta} e^{a(s)} ds,$$

which can be rewritten as

$$M(r)r = V'_b(0) \frac{\sigma^2}{2\delta} \int_r^\infty e^{a(s)-a(r)} ds. \quad (6)$$

Denoting  $I(r) = \int_r^\infty e^{a(s)-a(r)} ds$ , we will prove that  $I(r)$  is finite for enough large  $r$ . Noting that there exists  $r_1 > 0$  s.t.

$$-\frac{\sigma^2}{2\delta}t - \frac{\mu}{\delta} + \frac{-\lambda \widehat{P}(t) + \rho + \lambda}{t\delta} \leq -\frac{\sigma^2}{2\delta}t \quad t \geq r_1,$$

we have

$$a(s) - a(r) \leq \int_r^s -\frac{\sigma^2}{2\delta}t dt = -\frac{\sigma^2}{4\delta}(s^2 - r^2) \quad r \geq r_1.$$

Since there exists  $r_2 > 0$  s.t.  $e^{-\frac{\sigma^2}{4\delta}s^2} \leq \frac{1}{\frac{\sigma^2}{4\delta}s^2}$  for  $s \geq r_2$ , we obtain

$$I(r) \leq e^{\frac{\sigma^2}{4\delta}r^2} \int_r^\infty e^{-\frac{\sigma^2}{4\delta}s^2} ds \leq \frac{4\delta}{\sigma^2} e^{\frac{\sigma^2}{4\delta}r^2} \int_r^\infty \frac{1}{s^2} ds = \frac{4\delta}{\sigma^2} e^{\frac{\sigma^2}{4\delta}r^2} \frac{1}{r}, \quad r \geq r_1 \vee r_2 \vee 1.$$

This proves the finiteness of  $I(r)$ .

If  $V'_b(0) = 0$  held, then we would have  $M(r) = 0$  for all  $r \geq r_1 \vee r_2 \vee 1$  from the equation (6). This contradicts the definition of  $M(r)$ . If  $V'_b(0) = \infty$  held, then we would get  $M(r)r = \infty$  for all  $r > 1$  as  $I(r) > 0$ , which is contrary to the fact that

$$\lim_{r \rightarrow \infty} M(r)r = \lim_{r \rightarrow \infty} \int_0^\infty e^{-rx} r V_b(x) dx \leq \lim_{r \rightarrow \infty} \int_0^\infty \frac{r}{e^{rx}} [V_b(b) + x] dx = 0,$$

where the second inequality holds from (4). We have thus proved the theorem.

III. MAIN RESULTS

In this section, we first solve equation (2) on  $(0, \infty)$  with initial conditions  $g(0) = 0$  and  $g'(0) = C$ . Then we prove this solution on  $(0, b)$  is just  $V_b(x)$ , and give the optimal value of dividend barrier.

Equation (2) is an integro-differential equation. Differentiating equation and Laplace transforms are common tools to solve such problems. However, they can not be applied to our model. Differentiating equation can easily get rid of integral operator only for special distributions, e.g., exponential distribution; Laplace transforms are suitable for the case of constant coefficients. We instead transform the integro-differential equation to an integral equation, which is a convenient and practically useful way.

Integrating (2) over  $x$  twice and interchanging the order of integration, we have

$$\frac{1}{2}\sigma^2 g(x) = \frac{1}{2}\sigma^2 xg'(0) + \int_0^x g(t)k(x, t)dt, \quad x \geq 0, \quad (7)$$

where

$$k(x, t) = \mu + (2\delta + \lambda + \rho)t + x(-\delta - \lambda - \rho) + \lambda \int_t^x p(y-t)(x-y)dy, \quad 0 \leq t \leq x.$$

Equation (7) is a linear Volterra equation of the second kind. According to the theory on Volterra equation (e.g., [2], Section 2.1), equation (7) has a unique solution under initial conditions  $g(0) = 0$  and  $g'(0) = C$ . Using successive approximation method, we obtain

$$g(x) = C \left\{ x + \int_0^x \sum_{n=1}^{\infty} \left(\frac{-2}{\sigma^2}\right)^n k_n(x, t) dt \right\}, \quad x \geq 0, \quad (8)$$

where

$$k_n(x, t) = \int_t^x k(x, \bar{u}) k_{n-1}(\bar{u}, t) d\bar{u}, \quad 0 \leq t \leq x,$$

and we note that  $g(x)$  converge uniformly for any parameters.

We determine  $C$  by condition  $g'(b) = 1$ . Taking derivatives in (7), we get

$$g'(x) = C - C \frac{2}{\sigma^2} (\mu + \delta x)x + C \int_0^x \sum_{n=1}^{\infty} \left(\frac{-2}{\sigma^2}\right)^n \frac{\partial k_n}{\partial x}(x, t) dt, \quad x \geq 0$$

where

$$\begin{aligned} \frac{\partial k_1}{\partial x}(x, t) &= -\delta - \lambda - \rho, \quad 0 \leq t \leq x, \\ \frac{\partial k_n}{\partial x}(x, t) &= (\mu + \delta x) k_{n-1}(x, t) \\ &+ \int_t^x [-\delta - \rho - \lambda + \lambda P(x - \bar{u})] k_{n-1}(\bar{u}, t) d\bar{u} \\ & \quad n \geq 2, \quad 0 \leq t \leq x. \end{aligned}$$

Therefore, we have

$$C = \left[ 1 - \frac{2}{\sigma^2} (\mu + \delta b)b + \int_0^b \sum_{n=1}^{\infty} \left(\frac{-2}{\sigma^2}\right)^n \frac{\partial k_n}{\partial x}(b, t) dt \right]^{-1}.$$

Now we can get the formula of the dividend payments.

**Theorem 3.1:** If the density function  $p(x)$  of the claim sizes is continuous, the dividend payments  $V_b(x)$  has the following structure:

$$V_b(x) = \begin{cases} g(x) & 0 \leq x < b, \\ x - b + g(b) & x \geq b. \end{cases}$$

where  $g(x)$  is given by (8).

*Proof:* The result will be proved if  $g(x)$  satisfies conditions of Theorem 2.1. It is easy to check that  $g(x)$  satisfies equation (2) and boundary conditions (3) on  $(0, b)$ . In fact, we can also see that  $g(x)$  is twice continuously differentiable according to the theory on existence and uniqueness of integro-differential equations (see, e.g., [10], Theorem 3.14) under the continuity assumption on  $p(x)$ . Furthermore, we know that  $g(x)$  and  $g'(x)$  is bounded when  $x \in [0, b]$ . This completes the proof.

Let

$$m(x) = x + \int_0^x \sum_{n=1}^{\infty} \left(\frac{-2}{\sigma^2}\right)^n k_n(x, t) dt, \quad x \geq 0,$$

$$C(x) = \left[ 1 - \frac{2}{\sigma^2} (\mu + \delta x)x + \int_0^x \sum_{n=1}^{\infty} \left(\frac{-2}{\sigma^2}\right)^n \frac{\partial k_n}{\partial x}(x, t) dt \right]^{-1} \quad (9)$$

From Theorem 3.1 we can write  $V_b(x)$  as  $V_b(x) = c(b)m(x)$ , which means the dividend payments vary with change of barriers even with the same initial surplus. From this fact, we have the following corollary.

**Corollary 3.2:** The optimal value of dividend barrier  $b^*$  can be obtained as a solution of the equation  $C'(x) = 0$ , where  $C(x)$  is given by (9).

**Remark 1:** Although formulations (8) and (9) are not on closed form, they are still of important practical interest. Since  $g(x)$  converges uniformly for any parameters, current numerical methods of Volterra equation can work well for approximate solution.

#### IV. NUMERICAL EXAMPLES

In the following, we consider two numerical examples, which come from the two main classes of claim size distributions: exponentially decreasing tail and subexponential distributions.

**Example 1:** For an exponential claim size distribution, we assume that  $p(y) = \beta e^{-\beta y}$ . Let  $\rho = 0.05$ ,  $\mu = 1.1$ ,  $\lambda = 1$ ,  $\beta = 1$  and step size  $h = 0.005$  be fixed. The results are shown in Figure 1 and 2.

In Figure 1, we let  $\sigma = 0.5$  and  $\delta = 0$ . The optimal value of dividend barrier after calculation is  $b^* = 0.8305$ , which is consistent with the result of [8]. We calculate the value of dividend payments in the presence of dividend barrier  $b^* = 0.8305$ ,  $b = 0.5$  and  $b = 10$  respectively. Figure 1 shows the comparison of the results, from which we can see the optimality of the dividend barrier  $b^*$ , as the value of its dividend payments is greater than the other two.

In Figure 2, we let  $\sigma = 1$ , and  $\delta$  take values of 0, 0.01 and 0.02. The value of optimal dividend barriers for three cases are  $b_1^* = 1.69375$ ,  $b_2^* = 1.8225$  and  $b_3^* = 2.03375$ . Figure 2 shows their corresponding optimal dividend payments, from which we conclude that as interest rate rises, the value of optimal dividend payments increases.

**Example 2:** For a Pareto claim size distribution, we assume that  $p(y) = \frac{\alpha \beta^\alpha}{(x + \beta)^{\alpha + 1}}$ . Let  $\rho = 0.05$ ,  $\mu = 1.2$ ,  $\lambda = 1$ ,  $\alpha = 3$ ,  $\beta = 2$  and  $h = 0.005$  be fixed. The results are shown in Figure 3 and 4.

In Figure 3, we let  $\sigma = 1$  and  $\delta = 0.01$ . The optimal value of dividend barrier after calculation is  $b^* = 3.53$ . We calculate the value of dividend payments in the presence of dividend barrier  $b^* = 3.53$ ,  $b = 1$  and  $b = 6$  respectively. Figure 3 shows the comparison of the results, from which the optimality of the dividend barrier  $b^*$  can be also seen.

In Figure 4, we let  $\delta = 0.01$ , and  $\sigma$  take values of 0.6, 0.8 and 1. The value of optimal dividend barriers for three cases are  $b_4^* = 3.0025$ ,  $b_5^* = 3.24625$  and  $b_6^* = 3.53$ . Figure 4 shows their corresponding optimal dividend payments, from which we conclude that as volatility increases, the optimal dividend payments decreases. One possible explanation is that if volatility is larger, ruin occurs sooner in some sense.

#### V. CONCLUSIONS

In this paper, we consider a dividend problem for an insurance company in the presence of a dividend barrier with

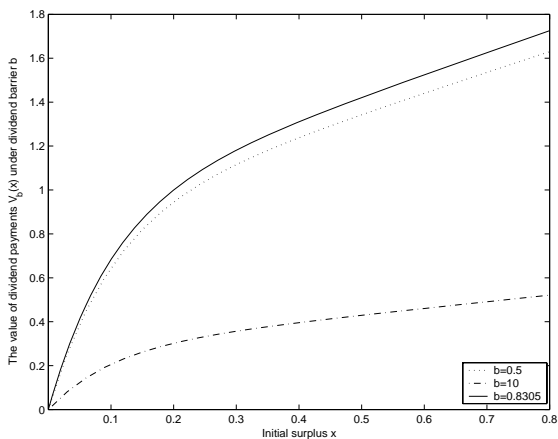


Fig. 1. Dividend payments under barrier  $b$ , exponential claims

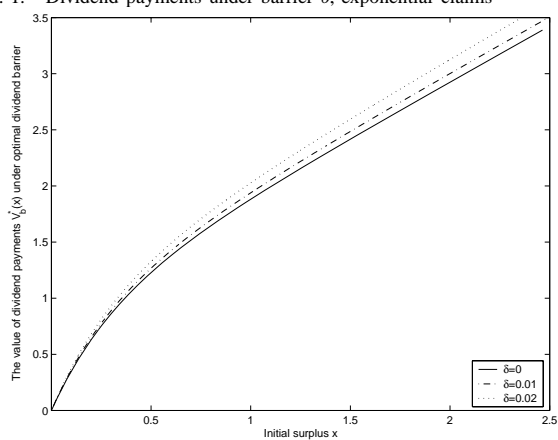


Fig. 2. Impact of changing the interest rate  $\delta$

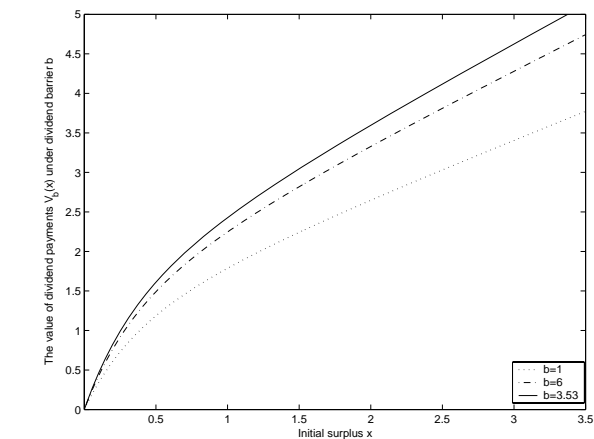


Fig. 3. Dividend payments under barrier  $b$ , pareto claims

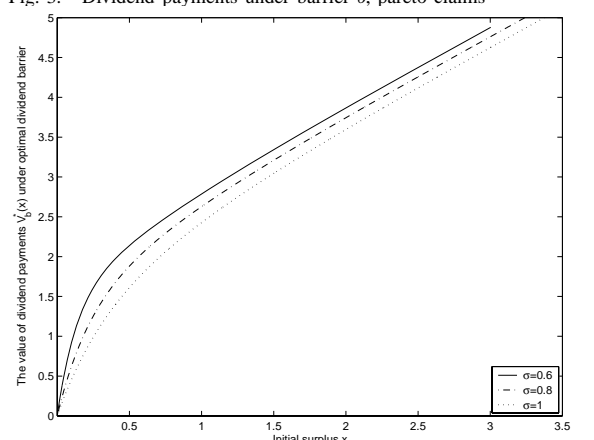


Fig. 4. Impact of changing the volatility  $\sigma$

constant interest rate. We formulate the total expected present value of dividends for general claim size distributions and derive the optimal value of dividend barrier. Our numerical results show consistency with daily views that dividend payments increase with interest rates, but decrease with the volatility.

Compared to the existing formula of dividend payments, our result has broader application as it can be used for general claim size distributions. It would also be interesting to study the case of stochastic return on interest rate. Although we expect that the analysis would be similar, we speculate that the solution structure would be more complicated.

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