

Certain Subordination Results For A Class Of Analytic Functions Defined By The Generalized Integral Operator

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Abstract—We obtain several interesting subordination results for a class of analytic functions defined by using a generalized integral operator.

Keywords—Analytic functions, Hadamard product, Subordinating factor sequence

I. INTRODUCTION

Let \mathcal{A} be the class of analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

defined on the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Let \mathcal{S} denote the subclass of \mathcal{A} consisting of functions that are univalent in \mathcal{U} . The Hadamard product of two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in \mathcal{A} is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad (2)$$

For complex parameters $\alpha_1, \dots, \alpha_q$ and β_1, \dots, β_s ($\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- = 0, -1, -2, \dots; j = 1, \dots, s$), we define the generalized hypergeometric function ${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ by

$${}_qF_s(\alpha_1, \alpha_2, \dots, \alpha_q; \beta_1, \beta_2, \dots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n z^n}{(\beta_1)_n \dots (\beta_s)_n n!}$$

$$(q \leq s + 1; q, s \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}; z \in \mathcal{U}),$$

where \mathcal{N} denotes the set of positive integers and $(x)_k$ is the Pochhammer symbol defined,

$$(x)_k = \begin{cases} 1 & \text{if } k = 0 \\ x(x+1) \dots (x+k-1) & \text{if } k \in \mathcal{N} = \{1, 2, \dots\}. \end{cases}$$

Corresponding to a function $h(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ defined by

$$h(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = z {}_qF_s(\alpha_1, \alpha_2, \dots, \alpha_q; \beta_1, \beta_2, \dots, \beta_s; z) \quad (3)$$

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Dziok and Srivastava considered a linear operator

$$H(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) : \mathcal{A} \longrightarrow \mathcal{A}$$

defined by Hadamard product (or convolution):

$$H(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)f(z) = h(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) * f(z)$$

The linear operator $H(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)f(z)$ includes (as its special cases) various other linear operators which were introduced and studied by Hohlov [9], Carlson and Shaffer [2], Ruscheweyh [13] and so on.

Corresponding to the function $h(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ defined by (3), we define a function $h_{\mu, p}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ given by

$$z + \sum_{n=2}^{\infty} \frac{(n+\mu)^p}{(\mu+1)^p} z^n * h_{\mu, p}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = h(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \quad (4)$$

Analogous to $H(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$, Selvaraj et al. [14] considered a linear operator

$\mathcal{G}_{\mu}^p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ on \mathcal{A} as follows:

$$\mathcal{G}_{\mu}^p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)f(z) = h_{\mu, p}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) * f(z) \quad (5)$$

($\alpha_i, \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-; i = 1, \dots, q; j = 1, \dots, s; \mu \neq -1; p \in \mathcal{N}_0 = \{0, 1, 3, \dots\}$)

For convenience, we write

$$\mathcal{G}_{\mu, q, s}^p(\alpha_1) = \mathcal{G}_{\mu}^p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \quad (6)$$

It can be easily verified from the definition of (5) that

$$z(\mathcal{G}_{\mu, q, s}^{p+1}(\alpha_1)f(z))' = (\mu+1)\mathcal{G}_{\mu, q, s}^p(\alpha_1)f(z) - \mu\mathcal{G}_{\mu, q, s}^{p+1}(\alpha_1)f(z) \quad (7)$$

and

$$z(\mathcal{G}_{\mu, q, s}^p(\alpha_1)f(z))' = \alpha_1\mathcal{G}_{\mu, q, s}^p(\alpha_1+1)f(z) - (\alpha_1-1)\mathcal{G}_{\mu, q, s}^p(\alpha_1)f(z) \quad (8)$$

we now define the most generalized subclass of \mathcal{A} by using the operator (5).

Definition 1.1. For any non-zero complex number λ , $0 \leq \gamma < 1$ and $k \geq 0$, a function $f \in \mathcal{A}$ is said to be in $\mathcal{I}(\alpha_1, q, s : k, \gamma, \lambda)$ if and only if it satisfies the condition

$$\Re \left\{ \frac{\lambda \mathcal{G}_{\mu, q, s}^p(\alpha_1 + 1)f(z)}{\mathcal{G}_{\mu, q, s}^p(\alpha_1)f(z)} - (\lambda - 1) \right\} > k \left| \frac{\lambda \mathcal{G}_{\mu, q, s}^p(\alpha_1 + 1)f(z)}{\mathcal{G}_{\mu, q, s}^p(\alpha_1)f(z)} - \lambda \right| + \gamma \quad (9)$$

The family $\mathcal{I}(\alpha_1, q, s : k, \gamma, \lambda)$ is of special interest for it contains many well-known as well as many new classes of analytic univalent functions. For $\lambda = \alpha_1$ and for appropriate choices of the parameters $\mathcal{I}(\alpha_1, q, s : k, \gamma, \lambda)$ reduces to $\mathcal{L}(a, c; \alpha, \beta)$ [5]. We note that the family $\mathcal{S}^*(\alpha)$ of starlike function of order α ($0 \leq \alpha < 1$)[3], [6], the family $\mathcal{C}(\alpha)$ of convex function of order α ($0 \leq \alpha < 1$)[3], [6], $k - UCV(\alpha)$ [1], $k - UST(\alpha)$ and many other well known subclasses of \mathcal{S} (see also the work of Kanas and Srivastava [10], Goodman [7], [8] and Rønning [11], [12]) are the special cases of $\mathcal{I}(\alpha_1, q, s : k, \gamma, \lambda)$.

We now state the definitions and lemmas which we need in the sequel.

Definition 1.2. An analytic function g is said to be subordinate to an analytic function f if $g(z) = f(w(z))$, $z \in \mathcal{U}$ for some analytic function w with $|w(z)| \leq |z|$ and we write $f(z) \prec g(z)$

Definition 1.3. A sequence $\{b_n\}_{n=1}^\infty$ of complex numbers is called a subordinating factor sequence if, whenever $f(z)$ is analytic, univalent and convex in \mathcal{U} , we have the subordination given by

$$\sum_{n=1}^\infty b_n a_n z^n \prec f(z) \quad (z \in \mathcal{U}, a_1 = 1). \quad (10)$$

Lemma 1.1[16]The sequence $\{b_n\}_{n=1}^\infty$ is a subordinating factor sequence if and only if

$$\Re \left\{ 1 + 2 \sum_{n=1}^\infty b_n z^n \right\} > 0 \quad (z \in \mathcal{U}). \quad (11)$$

Lemma 1.2 If

$$\sum_{n=2}^\infty \left(\lambda(1+k)(\alpha_1+1)_{n-1} + [(1-\gamma) - \lambda(1+k)](\alpha_1)_{n-1} \right) \Psi(n) |a_n| \leq (1-\gamma) \quad (12)$$

where $\Psi(n) = \frac{(\mu+1)^p(\alpha_2)_{n-1} \dots (\alpha_q)_{n-1}}{(n+\mu)^p(\beta_1)_n \dots (\beta_s)_n}$,

then $f(z) \in \mathcal{I}(\alpha_1, q, s : k, \gamma, \lambda)$

Proof. It suffices to show that

$$k \left| \frac{\lambda \mathcal{G}_{\mu, q, s}^p(\alpha_1 + 1)f(z)}{\mathcal{G}_{\mu, q, s}^p(\alpha_1)f(z)} - \lambda \right| - \Re \left\{ \frac{\lambda \mathcal{G}_{\mu, q, s}^p(\alpha_1 + 1)f(z)}{\mathcal{G}_{\mu, q, s}^p(\alpha_1)f(z)} - \lambda \right\} \leq 1 - \gamma.$$

We have

$$k \left| \frac{\lambda \mathcal{G}_{\mu, q, s}^p(\alpha_1 + 1)f(z)}{\mathcal{G}_{\mu, q, s}^p(\alpha_1)f(z)} - \lambda \right|$$

$$\begin{aligned} & - \Re \left\{ \frac{\lambda \mathcal{G}_{\mu, q, s}^p(\alpha_1 + 1)f(z)}{\mathcal{G}_{\mu, q, s}^p(\alpha_1)f(z)} - \lambda \right\} \\ & \leq (1+k) \left| \frac{\lambda \mathcal{G}_{\mu, q, s}^p(\alpha_1 + 1)f(z)}{\mathcal{G}_{\mu, q, s}^p(\alpha_1)f(z)} - \lambda \right| \\ & \leq \frac{(1+k) \sum_{n=2}^\infty \lambda(\alpha_1+1)_{(n-1)} \Psi(n) |a_n| |z|^{n-1}}{1 - \sum_{n=2}^\infty (\alpha_1)_{n-1} \Psi(n) |a_n| |z|^{n-1}} \\ & \quad - \frac{\lambda(\alpha_1)_{(n-1)} \Psi(n) |a_n| |z|^{n-1}}{1 - \sum_{n=2}^\infty (\alpha_1)_{n-1} \Psi(n) |a_n| |z|^{n-1}} \\ & \leq \frac{(1+k) \sum_{n=2}^\infty \lambda(\alpha_1+1)_{(n-1)} \Psi(n) |a_n| |z|^{n-1}}{1 - \sum_{n=2}^\infty (\alpha_1)_{n-1} \Psi(n) |a_n| |z|^{n-1}} \\ & \quad - \frac{\lambda(\alpha_1)_{(n-1)} \Psi(n) |a_n| |z|^{n-1}}{1 - \sum_{n=2}^\infty (\alpha_1)_{n-1} \Psi(n) |a_n| |z|^{n-1}} \end{aligned}$$

The last expression is bounded by $1 - \gamma$ if

$$\begin{aligned} & \sum_{n=2}^\infty \left(\lambda(1+k)(\alpha_1+1)_{n-1} \right. \\ & \left. + [(1-\gamma) - \lambda(1+k)](\alpha_1)_{n-1} \right) \Psi(n) |a_n| \leq (1-\gamma) \end{aligned}$$

which completes the proof of the Lemma.

For convenience we shall henceforth denote

$$\xi_n(\alpha_1, q, s : k, \gamma, \lambda) = \quad (13)$$

$$(\lambda(1+k)(\alpha_1+1)_{n-1} + [(1-\gamma) - \lambda(1+k)](\alpha_1)_{n-1}) \Psi(n)$$

Let $\mathcal{I}^*(\alpha_1, q, s : k, \gamma, \lambda)$ denote the class of functions $f(z) \in \mathcal{A}$ whose coefficients satisfy the conditions (12). We note that $\mathcal{I}^*(\alpha_1, q, s : k, \gamma, \lambda) \subseteq \mathcal{I}(\alpha_1, q, s : k, \gamma, \lambda)$.

II. MAIN RESULTS

we begin with the following:

Theorem 2.1 Let the function $f(z)$ defined by (1) be in the class $\mathcal{I}^*(\alpha_1, q, s : k, \gamma, \lambda)$ where $0 \leq \gamma < 1$; $k \geq 0$. Also let \mathcal{C} denote the familiar class of functions $f(z) \in \mathcal{A}$ which are univalent and convex in \mathcal{U} . Then

$$\frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{2[1-\gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)]} (f * g)(z) \prec g(z) \quad (14)$$

and ($z \in \mathcal{U}$; $g \in \mathcal{C}$)

$$\Re(f(z)) > - \frac{1-\gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)} \quad (15)$$

The constant $\frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{2[1-\gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)]}$ is the best estimate.

Proof. Let $f(z) \in \mathcal{I}^*(\alpha_1, q, s : k, \gamma, \lambda)$ and let $g(z) = z + \sum_{n=2}^\infty b_n z^n \in \mathcal{C}$. Then

$$\begin{aligned} & \frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{2[1-\gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)]} (f * g)(z) = \\ & \frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{2[1-\gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)]} \left(z + \sum_{n=2}^\infty a_n b_n z^n \right). \end{aligned}$$

Thus, by Definition (1.3), the assertion of the theorem will hold if the sequence

$$\left\{ \frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{2[1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)]} a_n \right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma (1.1), this will be true if and only if ($z \in \mathcal{U}$)

$$\Re \left\{ 1 + 2 \sum_{n=1}^{\infty} \frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{2[1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)]} a_n z^n \right\} > 0. \tag{16}$$

Now

$$\begin{aligned} & \Re \left\{ 1 + \frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)} \sum_{n=1}^{\infty} a_n z^n \right\} \\ &= \Re \left\{ 1 + \frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)} z + \right. \\ & \quad \left. \frac{1}{1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)} \sum_{n=2}^{\infty} \xi_2(\alpha_1, q, s : k, \gamma, \lambda) a_n z^n \right\} \\ & \geq 1 - \left\{ \frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)} r - \right. \\ & \quad \left. \frac{1}{1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)} \sum_{n=2}^{\infty} \xi_2(\alpha_1, q, s : k, \gamma, \lambda) a_n r^n \right\}. \end{aligned}$$

Since $\xi_n(\alpha_1, q, s : k, \gamma, \lambda)$ is an increasing function of n ($n \geq 2$)

$$\begin{aligned} & 1 - \left\{ \frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)} r - \right. \\ & \quad \left. \frac{1}{1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)} \sum_{n=2}^{\infty} \xi_2(\alpha_1, q, s : k, \gamma, \lambda) a_n r^n \right\} \\ & > 1 - \frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)} r \\ & \quad - \frac{1 - \gamma}{1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)} r > 0. \end{aligned}$$

Thus (16) holds true in \mathcal{U} . This proves the inequality (14). The inequality (15) follows by taking the convex function $g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n$ in (14). To prove the sharpness of the constant $\frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{2[1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)]}$, we consider $f_0(z) \in \mathcal{I}^*(\alpha_1, q, s : k, \gamma, \lambda)$ given by

$$f_0(z) = z - \frac{1 - \gamma}{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)} z^2 \quad (-1 \leq \gamma < 1).$$

Thus from (14), we have

$$\frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{2[1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)]} f_0(z) \prec \frac{z}{1-z}. \tag{17}$$

It can be easily verified that

$$\min \left\{ \Re \left(\frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{2[1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)]} f_0(z) \right) \right\} = -\frac{1}{2}$$

This shows that the constant $\frac{\xi_2(\alpha_1, q, s : k, \gamma, \lambda)}{2[1 - \gamma + \xi_2(\alpha_1, q, s : k, \gamma, \lambda)]}$ is best possible. This completes the proof of the Theorem.

Let $\lambda = \alpha_1$ and for the appropriate choice of μ, p, q, s , we have the following:

Corollary 2.1.[4] Let $f(z)$ be defined by (1) be in the class $\mathcal{L}^*(a, c, k, \gamma)$ and satisfy the condition

$$\sum_{n=2}^{\infty} \left(\frac{(1+k)(\alpha_1)_n}{(\beta_1)_{n-1}} + \frac{[(1-\gamma) - \alpha_1(1+k)](\alpha_1)_{n-1}}{(\beta_1)_{n-1}} \right) |a_n| \leq (1-\gamma),$$

then for $z \in \mathcal{U}; g \in \mathcal{C}$,

$$\frac{\sigma_2(\alpha_1, \beta_1 k, \gamma)}{2[1 - \gamma + \sigma_2(\alpha_1, \beta_1 k, \gamma)]} (f * g)(z) \prec g(z), \tag{18}$$

and

$$\Re(f(z)) > -\frac{1 - \gamma + \sigma_2(\alpha_1, \beta_1 k, \gamma)}{\sigma_2(\alpha_1, \beta_1 k, \gamma)} \tag{19}$$

where

$$\sigma_2(\alpha_1, \beta_1 k, \gamma) = \frac{(1+k)(\alpha_1)_n + [(1-\gamma) - \alpha_1(1+k)](\alpha_1)_{n-1}}{(\beta_1)_{n-1}}.$$

The constant $\frac{\sigma_2(\alpha_1, \beta_1 k, \gamma)}{2[1 - \gamma + \sigma_2(\alpha_1, \beta_1 k, \gamma)]}$ is the best estimate.

Corollary 2.2. Let the function $f(z)$ be defined by (1) be starlike of order γ in \mathcal{U} and satisfy the condition

$$\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha, \text{ then}$$

$$\frac{2 - \alpha}{6 - 4\alpha} (f * g)(z) \prec g(z) \quad (z \in \mathcal{U}; g \in \mathcal{C}), \tag{20}$$

and

$$\Re(f(z)) > -\frac{3 - 2\alpha}{2 - \alpha}, \quad (z \in \mathcal{U}). \tag{21}$$

Corollary 2.3.[15] Let the function $f(z)$ be defined by (1) be starlike in \mathcal{U} and satisfy the condition $\sum_{n=2}^{\infty} n |a_n| \leq 1$, then

$$\frac{1}{3} (f * g)(z) \prec g(z) \quad (z \in \mathcal{U}; g \in \mathcal{C}) \tag{22}$$

and

$$\Re(f(z)) > -\frac{3}{2}, \quad (z \in \mathcal{U}). \tag{23}$$

Corollary 2.4. Let the function $f(z)$ be defined by (1) be convex in \mathcal{U} and satisfy the condition $\sum_{n=2}^{\infty} n^2 |a_n| \leq 1$, then

$$\frac{2}{5} (f * g)(z) \prec g(z) \quad (z \in \mathcal{U}; g \in \mathcal{C}) \tag{24}$$

and

$$\Re(f(z)) > -\frac{5}{4}, \quad (z \in \mathcal{U}). \tag{25}$$

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