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Winding Numbers of Paths of Analytic Functions Zeros in Finite Quantum Systems

Muna Tabuni

Abstract—The paper contains an investigation of winding numbers of paths of zeros of analytic theta functions. We have considered briefly an analytic representation of finite quantum systems Z_N. The analytic functions on a torus have exactly N zeros. The brief introduction to the zeros of analytic functions and there time evolution is given. We have discussed the periodic finite quantum systems. We have introduced the winding numbers in general. We consider the winding numbers of the zeros of analytic theta functions.

Keywords—Winding numbers, period, paths of zeros.

I. Introduction

THIS Paper is devoted to the study winding numbers of paths of zeros in analytic representation of finite quantum systems on a torus. Analytic functions are very important tool in several branches of physic sciences. [1], [2], [3] has been studied analytic functions and used them Widely in quantum mechanics. The analytic Bargmann function [4], [5], [6], [7], [8], [9], [10] is the important to study the overcompleteness of the coherent states. Ref [11], [12] have studied analytic representations of finite quantum systems on a torus. The analytic function representing a quantum state has exactly N zeros which define uniquely the quantum state. Ref [13] has been studied the motion of the N zeros on the torus. In present paper we introduce the winding numbers of the zeros of analytic functions. The path of zeros are functions of time. The path of this motion is curve as long as functions $x(t)$ and $y(t)$ are continuous. We define the winding numbers of real part and imaginary part of the zero. We demonstrate these general ideas with various concrete examples.

II. Analytic representation of finite quantum systems

Let $\mathbb{H}$ be a d-dimensional Hilbert space. Let $|X_m\rangle, |P_m\rangle$, where m the integers modulo n, be an orthonormal basis in this Hilbert space (position states and momentum states Respectively). where

$$|P_m\rangle = \mathbb{F}|X_m\rangle = N^{-1/2}\sum_n (\exp\left[i\frac{2\pi m}{N}\right])|X_m\rangle, \qquad (1)$$

and $\mathbb{F}$ the Fourier operator:

$$\mathbb{F} = N^{-1/2}\sum_{m,n}(\exp\left[i\frac{2\pi m}{N}\right])|X_m\rangle\langle X_n|. \qquad (2)$$

M. Tabuni is with the Department of Mathematics, University of Tripoli, Libya(e-mail: mona $\_bashat@yahoo.co.uk$).

Let x, p be the position and momentum operators and they given by

$$x = \sum_{n=0}^{N-1} n|X_n\rangle\langle X_n, \qquad (3)$$

$$p = \mathbb{F}x\mathbb{F}^{\dagger} = \sum_{n=0}^{N-1} n|P_n\rangle\langle P_n| \qquad (4)$$

We study an arbitrary normalized state $|\mathbb{F}\rangle$

$$|\mathbb{F}\rangle = \sum_m \mathbb{F}_m|X_m\rangle; \qquad \sum_m |\mathbb{F}_m|^2 = 1, \qquad (5)$$

By reference to ref[13] we represent the state $|\mathbb{F}\rangle$ of Eq.(5), with the analytic function

$$f(z) = \pi^{-1/4}\sum_{m=0}^{N-1}\mathbb{F}_m\vartheta_3[\pi m N^{-1} - \sqrt{\frac{z\pi}{2N}}; \frac{i}{N}] \qquad (6)$$

which obeys quasi-periodic relations

$$f\left[z+\sqrt{2\pi N}\right] = f(z)$$
$$f\left[z+i\sqrt{2\pi N}\right] = f(z)\exp\left[\pi N - i\sqrt{2\pi N}z\right], \qquad (7)$$

where ϑ_3 is Theta function defined as

$$\vartheta_3(u,\tau) = \sum_{n=-\infty}^{\infty}\exp(i\pi\tau n^2 + i2nu). \qquad (8)$$

The analytic function $f(z)$ is defined on a cell $[a, a+\sqrt{2\pi N})\times[b, b+\sqrt{2\pi N})$ (defined on a torus)

Example 1:
We consider the case where $N = 3$ and the state $|\mathbb{F}(0)\rangle$ at $t = 0$ is described through the coefficients

$$\mathbb{F}_0(0) = 0.08 - 0.24i,\ \mathbb{F}_1(0) = 0.52 + 0.45i,$$
$$\mathbb{F}_2(0\ \ ) = 0.55 + 0.37i. \qquad (9)$$

In Fig.1 we plot the real part of the function $f(z)$in Eq.(6)

III. Zeros of the functions $f(z)$

Ref.[12] has proved that the sum of the zeros μ_n of $f(z)$, is

$$\sum_{n=1}^{N}\mu_n = (2\pi)^{1/2}N^{3/2}(l+ir) + \left(\frac{\pi}{2}\right)^{1/2}N^{3/2}(1+i) \quad (10)$$

By reference to ref.[12], [13] we construct the function $f(z)$ from its zeros μ_n which satisfy the relation of Eq.(10) as

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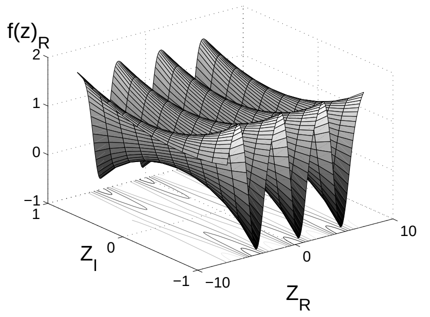

Fig. 1. The real part of the function $f(z)$ in Eq.(6) where $N=3$ and the $|F(t)\rangle$ at $t=0$ is described through the coefficients in Eq.(13).

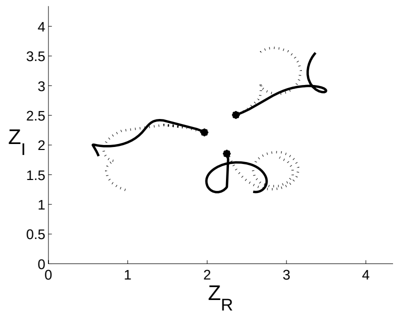

Fig. 2. The distribution of the zeros $\mu_n(t)$ for the state $|\mathbb{F}(t)\rangle$ which at $t=0$ is described in Eq.(13) for Hamiltonian H_1 (dotted line)and H_2 (solid line) of Eq.(14).

following

$$f(z) = q\exp\left[-i\left(\frac{2\pi}{N}\right)^{1/2} lz\right]\prod_{n=1}^{N}\vartheta_3\left[w_n(z);\ i\right]$$
$$w_n(z) = \left(\frac{\pi}{2N}\right)^{1/2}(z-\mu_n)+\frac{\pi(1+i)}{2} \quad (11)$$

where l is the integer relation of Eq.10; and q is a fixed calculated from the normalization condition.
Ref.[13] has calculated the coefficients $\mathbb{F}_m$from $f(z)$ as following.

IV. Paths of the zeros

Following ref.[13] we consider the state $|\mathbb{F}(0)\rangle = \sum \mathbb{F}_m(0)|X;m\rangle$ at $t=0$. Using the Hamiltonian H, the state $|\mathbb{F}(0)\rangle$ evolves at time t

$$|\mathbb{F}(t)\rangle = \exp(ith)|\mathbb{F}(0)\rangle = \sum_{m=0}^{N-1}\mathbb{F}_m(t)|X_m\rangle \quad (12)$$

Example 2:
We consider the case where $N=3$ and the state $|\mathbb{F}(0)\rangle$ at $t=0$ is described through the coefficients

$$\mathbb{F}_0(0) = 0.9-0.008i,\quad \mathbb{F}_1(0)=0.3+0.004i,$$
$$\mathbb{F}_2(0) = 0.3+0.003i. \quad (13)$$

We have calculated the coefficients $|\mathbb{F}(t)\rangle$ for the two cases of the Hamiltonians

$$H_1 = \frac{x^2}{2}+\frac{p^2}{2},$$
$$H_2 = -i\ln\left[\exp\left(\frac{ix^2}{2}\right)\exp\left(\frac{ip^2}{2}\right)\right]. \quad (14)$$

Using MATLAB we calculated numerically the zeros μ_n of $f(z)$. In Fig.2 we present the three curves μ_n for the Hamiltonian H_1 (dotted line), and the Hamiltonian H_2(sold line) of Eq.14.

V. Periodicity of the zeros

Ref.[13] has discussed the Periodic finite quantum systems. In some cases d of the zeros follow the same path. We say that this path has multiplicity d (see Ref.[13]).

Example 3:

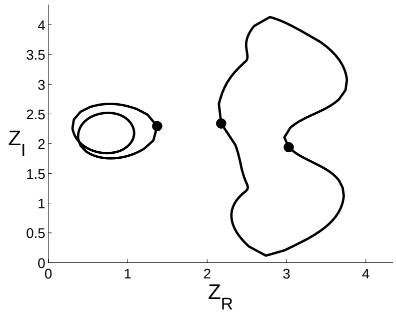

Fig. 3. The path of the zeros μ_0, μ_1, μ_2 with the Hamiltonian of Eq.16. The initial values of the zeros are given in Eq.15

Let

$$\mu_0(0) = 1.37+2.29i;\ \mu_1(0)=2.17+2.34i,$$
$$\mu_2(0) = 3.02+1.94i \quad (15)$$

be the zeros at $t=0$ and let

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (16)$$

be the Hamiltonian with eigenvalues $0,1,2$ with period $\alpha = 2\pi$. Numerically we get that

$$\mu_1(\alpha+t)=\mu_2(t),\quad \mu_2(\alpha+t)=\mu_1(t), \quad (17)$$

In this case after period the μ_1,μ_2 follow the same path and after a period they exchange position while μ_3 follows a closed path as following

$$\mu_1(\alpha)=\mu_2(0),\ \ \mu_2(\alpha)=\mu_1(0),\ \ \mu_0(\alpha)=\mu_0(0), \quad (18)$$

In Fig.3 we plot the paths of this zeros.

Example 4:
Let

$$\mu_0(0) = 0.7+2.6i,\ \mu_1(0)=2.1+4.3i,$$
$$\mu_2(0) = 3.7+1.1i,\ \mu_3(0)=3.7+2.2i, \quad (19)$$

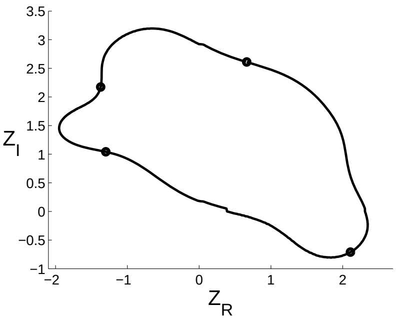

Fig. 4. The path of the zeros $\mu_0, \mu_1, \mu_2, \mu_3$ for the Hamiltonian of Eq.20. The initial values of the zeros are given in Eq. 19

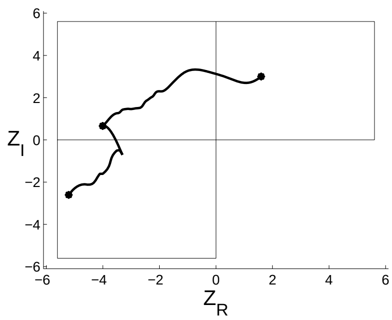

Fig. 5. The path of the zeros μ_0 with the Hamiltonian of Eq.14. The initial values of the zeros $\mu_0, \mu_1, \mu_2, \mu_3, \mu_4$ are given in Eq.25.

be the zeros at $t = 0$ and let

$$H = \begin{bmatrix} 1 & -i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad (20)$$

be the Hamiltonian with eigenvalues $0, 2, 2, 2$. Using Matlab we calculate the paths of the zeros and found that

$$\begin{aligned} \mu_0(\alpha + t) &= \mu_1(t),\ \mu_1(\alpha + t) = \mu_2(t), \\ \mu_2(\alpha + t) &= \mu_3(t),\ \mu_3(\alpha + t) = \mu_0(t) \end{aligned} \quad (21)$$

In this example all the zeros follow the same path and after period $\alpha = \pi$ we get that

$$\begin{aligned} \mu_0(\alpha) &= \mu_1(0),\ \mu_1(\alpha) = \mu_2(0), \\ \mu_2(\alpha) &= \mu_3(0),\ \mu_3(\alpha) = \mu_0(0). \end{aligned} \quad (22)$$

In Fig.4 we plot the paths of this zeros. Hence by definition the zeros μ_1, μ_2 in example.3 have multiplicity $d = 2$ and the four zeros in example.4 have multiplicity $d = 4$.

VI. Winding numbers of Paths of the zeros

The analytic function $f(z)$ in Eq.(6) obeys quasi-periodic relations of Eq.(7). In some cases the paths of the zeros are closed curves.
In this section we study the winding number of the paths of the zeros of analytic function $f(z)$.
In general the winding number of the closed curve C about a point z_0 is the number of times C surrounds z_0. In our case the analytic function $f(z)$ is defined on a cell
$[a, a + \sqrt{2\pi N}) \times [b, b + \sqrt{2\pi N})$,
and each cell is labeled by a two of integers (l, r) with area $2\pi N$ and period $\sqrt{2\pi N}$.
The paths of zeros are functions of time, so we can write each zero as following

$$\mu(t) = x(t) + iy(t), \quad 0 \leq t \leq T, \quad (23)$$

The path of this motion is curve as long as functions $x(t)$ and $y(t)$ are continuous.
Our goal is to calculate winding number of the paths μ_n.
We say that x have completed one period at $t = T$, if it obeys the relation
$x(T) = x(0) + \sqrt{2\pi N}$,
and we say that x have completed κ periods at $t = T$, if it obeys the relation
$x(T) = x(0) + \kappa\sqrt{2\pi N}$.
Here κ is the winding number of x.
We will denote a winding number of real part of the path of the zeros μ_n by κ and we will denote a winding number of imaginary part of the path of the zeros μ_n by η. Therefore

$$\kappa = \frac{x(T) - x(0)}{\sqrt{2\pi N}}, \quad \eta = \frac{y(T) - y(0)}{\sqrt{2\pi N}} \quad (24)$$

We present the following examples
Example 5:
Let

$$\begin{aligned} \mu_0(0) &= 2.8 + 1.8i,\ \mu_1(0) = 0.3 + 0.9,\ \mu_2(0) = 1.6 + 3i, \\ \mu_3(0) &= 3.93 + 4.8i,\ \mu_4(0) = 5.3 + 3.5i \end{aligned} \quad (25)$$

be the zeros at $t = 0$.
We consider the Hamiltonian H_1 in Eq.(14) for the case $N = 5$ which has the eigenvalues $12.82, 8.15, 5.17, 2.87, 0.96$.
We consider the zero $\mu_2(0) = 1.6 + 3i$.
When the system evolves in time, the zeros move in paths on the torus.
In this case we found numerically that

$$\begin{aligned} \mu_2(1.7) &= -4.006 + 0.66i = (1.6 - 5.605) + 0.66i \\ &= (1.6 + (-1)\sqrt{2\pi N}) + 0.66i \end{aligned} \quad (26)$$

This show that at the time $t = 1.7$ the winding number of the real part of the zero μ_2 is $\kappa = -1$. Therefore at the time t=1.7 the winding number of the of the zero μ_2 are $(\kappa, \eta) = (-1, 0)$.
At time $t = 3.5$ we found that

$$\begin{aligned} \mu_2(3.5) &= -5.205 - 2.605i = -5.205 - (3 - 5.605)i \\ &= -5.205 - (3 + (-1)\sqrt{2\pi N}i) \end{aligned} \quad (27)$$

It is seen that at the time $t = 3.5$ the winding number of the of the zero μ_2 are $(\kappa, \eta) = (-1, -1)$.
In Fig.5 we present the path of the zero μ_2.
Example 6:
Let

$$\begin{aligned} \mu_0(0) &= 0.5 + 3.8i,\ \mu_1(0) = 0.9 + 2.5i, \\ \mu_2(0) &= 0.93 + 0.3i, \end{aligned} \quad (28)$$

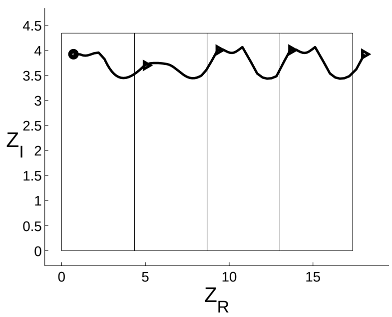

Fig. 6. The path of the zeros μ_0 with the Hamiltonian of Eq.14. The initial values of the zeros μ_0, μ_1, μ_2 are given in Eq.28.

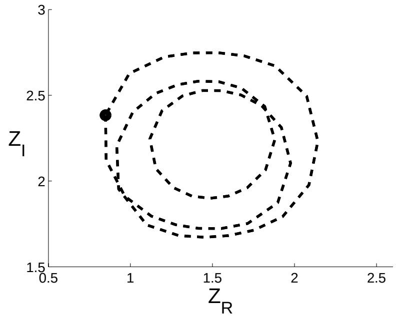

Fig. 7. The path of the zeros μ_1 with the Hamiltonian of Eq.14. The initial values of the zeros μ_0, μ_1, μ_2 are given in Eq.28.

be the zeros at $t = 0$,
and let

$$H = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad (29)$$

be the Hamiltonian with eigenvalues $-2, 2, 4$.
We consider the zero $\mu_0(0) = 0.44 + 3.76i$
In this case found numerically that

$$\begin{aligned} \mu_0(0.08) &= 0.7082 + \sqrt{2\pi N} + 3.698i, \\ \mu_0(0.17) &= 0.7082 + 2\sqrt{2\pi N} + 4.003i, \\ \mu_0(0.25) &= 0.7082 + 3\sqrt{2\pi N} + 4.003i. \end{aligned} \quad (30)$$

This show that at the times $t = 0.23,\ 0.4,\ 0.56$ the winding number (κ, η) of the path of the zero μ_0 are $(1,0),\ (2,0),\ (3.0)$ respectively
In Fig.6 we present the path of the zero μ_0.

Also we consider the zero

$$\mu_1(0) = 0.90 + 2.51i, \quad (31)$$

this zero comes to its original position at time $t = 0.5$ and we found that

$$\mu_1(0.5) = 0.90 + 0\sqrt{2\pi N} + 2.51i + 0\sqrt{2\pi N}. \quad (32)$$

Here we say that at the time $t = 0.5$ the winding number of the of the zero μ_1 are $(\kappa, \eta) = (0,0)$.
In Fig.7 we present the path of the zero μ_1.

VII. CONCLUSION

We have studied the analytic representation of finite quantum systems. The zeros of analytic theta function and there time evolution have been considered. We have derived some examples to calculate the paths of various zeros for various Hamiltonians. A brief discussion to the Periodicity of the zeros has been given. In some cases some of the zeros travel in one path.
In general the winding number of the closed curve C about a point z_0 is the number of times C surrounds z_0. We have introduced the definition of winding number of the zeros of the analytic representation in finite quantum systems.
The winding number of the zeros of analytic function $f(z)$ are expressed in integer pairs Eq.(24).
We gave several examples to calculate winding number of the paths $\mu_n(t)$ of various zeros.

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