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Comparison Analysis of the Wald's and the Bayes Type Sequential Methods for Testing Hypotheses

K. J. Kachiashvili

***Abstract*—**The Comparison analysis of the Wald's and Bayes-type sequential methods for testing hypotheses is offered. The merits of the new sequential test are: universality which consists in optimality (with given criteria) and uniformity of decision-making regions for any number of hypotheses; simplicity, convenience and uniformity of the algorithms of their realization; reliability of the obtained results and an opportunity of providing the errors probabilities of desirable values. There are given the Computation results of concrete examples which confirm the above-stated characteristics of the new method and characterize the considered methods in regard to each other.

***Keywords*—**Errors of types I and II, likelihood ratio, the Bayes Type Sequential test, the Wald's sequential test, averaged number of observations.

I. Introduction

THE development of sequential methods was started after the Second World War by Wald [1], [2] and Barnard [3]. In [4] was given a set of works dedicated to different aspects of the problem of the sequential analysis. The properties of optimality of the Wald criterion were investigated in [1], [2], [5]-[12]. In [8], [9] some modifications of Wald's method were developed. In particular, in [9] a modification which guaranteed achiving the exact error probabilities was developed. Sequential tests of the multidimensional type with the corresponding univariate sequential tests, with emphasis on the Gaussian setting, were compared in [13], [14]. For these cases the comparison of expected sample sizes is realized in [15]. The optimal properties of the Neyman-Pearson and Wald criteria were compared in [12]. There was shown that, for providing the given probabilities of errors of the first and the second kinds, in the Wald criterion, from half to one-third as many observation results as in the Neyman-Pearson criterion were needed in the case when the tested hypotheses were close.

The Bayesian sequential procedures were described in [10], [16]-[26] and others. The essence of these procedures consists in the minimization of the risk, which is defined as the average cost of observations plus the average loss resulting from erroneous decisions.

K. J. Kachiashvili is with Informatics and Control Systems Department, Georgian Technical University, 77, Kostava Street, 0175, Tbilisi, Georgia (e-mail: kartlos55@yahoo.com).

In [16], [26] and others it was shown that the Bayesian sequential procedures and the Wald criterion are optimum in the sense of definition of optimality in these criteria, and that under certain conditions, they coincide.

The methods of sequential analysis of the Wald's method and the method based on the Bayesian approach are quite simple, graphic and convenient for practical realization, but, unfortunately, only for the case of two hypotheses. For an arbitrary number of hypotheses, the problem becomes significantly complex, and it has not been solved completely in the sense of conventional statements of both the sequential criterion based on the sequential probability ratio test and the minimization of the sum of the Bayesian risk calculated for sequentially incoming observation results and the cost of obtaining the same results of the experiment.

New methods of sequential analysis for testing many hypotheses were offered in [27]; they are based on the specific properties of hypotheses acceptance regions in constrained Bayesian problems of testing many hypotheses [28]-[30]. The aim of this work is to provide the comparison analysis of new the Sequential and Wald's methods for testing hypotheses.

II. The Wald's Method

For the statement of the problem, let us use the Wald's formalization [1], [2]. Let H_0 and H_1 be the suppositions that a random variable X has the distribution density $p(x|H_1)$ or $p(x|H_2)$, respectively. The decision in favour of the hypothesis must be made on the basis of the sequentially obtained observation results $x_1, x_2, \ldots$. The essence of the Wald's sequential test consists in the following: to compute the likelihood ratio $B(\mathbf{x}) = p(x_1, x_2, \ldots, x_m | H_0) / p(x_1, x_2, \ldots, x_m | H_1)$ for m sequentially obtained observation results, and, if

$$B < B(\mathbf{x}) < A, \tag{1}$$

the decision is not made, and the observation of the random variable is continued. If

$$B(\mathbf{x}) \geq A, \tag{2}$$

then hypothesis H_0 is accepted on the basis of m observation results. If

$$B(\mathbf{x}) \leq B, \tag{3}$$

then hypothesis H_1 is accepted on the basis of m observation results.

The thresholds A and B are chosen so that the significance level and the power of the criterion are equal to α and $1-\beta$, respectively.

Finding of the exact values of A and B is a challenge. Therefore, for practical aims, their upper and lower estimations are suggested [1], [2], [31], [32] respectively.

$$A=\frac{1-\beta}{\alpha} \quad \text{and} \quad B=\frac{\beta}{1-\alpha}, \tag{4}$$

It is proved [1] that in this case the real values of the errors of types I and II are close enough to the desired values, but on the whole are distinguished from them.

As was mentioned above, unfortunately, the generalization of this method for an arbitrary number of hypotheses has not been accomplished.

III. A New Method of Sequential Analysis

In [33] new forms of the Bayesian statement of hypotheses testing were introduced. Instead of the unconstrained problem of minimization of the average risk caused by the errors of types I and II, it was offered to solve the constrained optimization problem. In this case, restrictions are imposed on the errors of one type and the errors of the second type are minimized. Depending on the type of restrictions, there are considered different constrained optimization problems [28], [33]: 1. The Restriction on the averaged probability of acceptance of true hypothesis; 2. The Restrictions on conditional probabilities of acceptance of each true hypothesis; 3. Restrictions on the posterior probabilities of acceptance of each true hypothesis; 4. The Restriction on the averaged probability of rejection of true hypotheses; 5. Restrictions on the probabilities of rejection of each true hypothesis; 6. Restrictions on the posteriori probabilities of rejection of each true hypothesis; 7 Restrictions on averaged probabilities of rejected true hypotheses. To be specific, let us consider the task of imposing the restriction on the averaged probability of rejection of true hypotheses, which has the following statement

$$\sum_{i=1}^{S} p(H_i)\int_{\Gamma_i} p(\mathbf{x}\,|\,H_i)dx \Rightarrow \max_{\{\Gamma_i\}}, \tag{5}$$

subject to

$$\sum_{i=1}^{S} p(H_i)\sum_{j=1,\,j\neq i}^{S}\int_{\Gamma_j} p(\mathbf{x}\,|\,H_i)dx \le \alpha. \tag{6}$$

The Solution of this problem is

$$\Gamma_j=\left\{x: p(H_j)\,p(\mathbf{x}\,|\,H_j) > \lambda\sum_{i=1,i\neq j}^{S} p(H_i)\,p(\mathbf{x}\,|\,H_i)\right\}, \quad j=1,..,S. \tag{7}$$

Here S is the number of tested hypotheses; H_i $(i=1,...,S)$ is the tested hypothesis; Γ_i is the region of acceptance of hypothesis H_i; $p(H_i)$ is the a priori probability of H_i hypothesis; $p(\mathbf{x}\,|\,H_i)$ is the conditional distribution density of the observation vector; λ is defined so that equality was fulfilled in (6).

The results of investigation of hypotheses acceptance regions (7) show that the decision-making space contains hypotheses acceptance regions and a no-decision region [28], [29]. This property is used for the introduction of a new sequential method of statistical hypotheses testing. The essence of the method is in the following [27].

Let us designate: Γ_i^m is the acceptance region of H_i hypotheses (7) on the basis of m sequentially obtained repeated observation results; R_m^n is the decision-making space in the sequential method; n is the dimensionality of the observation vector; I_i^m is the population of sub-regions of intersections of acceptance regions of hypotheses H_i, Γ_i^m $(i=1,...,S)$, with the regions of acceptance of other hypotheses H_j, $j=1,...,S$, $j\neq i$; $E_m^n=R_m^n-\bigcup_{i=1}^{S}\Gamma_i^m$ is the population of regions of space R_m^n which do not belong to any of hypotheses acceptance regions.

The hypotheses acceptance regions in the sequential method are:

$$R_{m,i}^n=\Gamma_i^m/I_i^m, \quad i=1,...,S; \tag{8}$$

the no-decision region is:

$$R_{m,S+1}^n=\left(\bigcup_{i=1}^{S} I_i^m\right)\bigcup E_m^n, \tag{9}$$

where the acceptance region of the H_i hypotheses

$$\Gamma_i^m=\{\mathbf{x}: p(\mathbf{x}\,|\,H_i) > \textstyle\sum_{\ell=1,\ell\neq i}^{S}\lambda_\ell^i\, p(\mathbf{x}\,|\,H_\ell)\}, \tag{10}$$

where $0\le\lambda_\ell^i<+\infty$, $\ell=1,...,S$.

Coefficients $\lambda_\ell^i=\lambda\dfrac{p(H_\ell)}{p(H_i)}$ are defined from the equality in the suitable restrictions.

These methods, obtained for all possible constrained optimization problems (see above), are called *the sequential methods of Bayesian type* [27]. To be specific, further we will consider the task with restrictions on the averaged probability of acceptance of true hypotheses.

IV. Comparison Analysis

Let us investigate the ratio among the errors of types I and II in the Walds's and sequential Bayesian-type methods when the number of hypotheses is two. For simplicity, let us omit the indexes where this does not cause misunderstanding. For two hypotheses regions (10) takes the forms

$$\Gamma_0=\{\mathbf{x}: B(\mathbf{x})>\lambda\frac{p(H_0)}{p(H_1)}\} \text{ and } \Gamma_1=\{\mathbf{x}: B(\mathbf{x})<\frac{1}{\lambda}\frac{p(H_0)}{p(H_1)}\} \tag{11}$$

in the considered task with restrictions on the averaged probability of acceptance of true hypotheses, the no-decision region is: $\frac{1}{\lambda}\frac{p(H_0)}{p(H_1)}\le B(\mathbf{x})\le\lambda\frac{p(H_0)}{p(H_1)}\}$ when $\lambda\frac{p(H_0)}{p(H_1)}>1$ and $\lambda\frac{p(H_0)}{p(H_1)}\le B(\mathbf{x})\le\frac{1}{\lambda}\frac{p(H_0)}{p(H_1)}\}$ at $\lambda\frac{p(H_0)}{p(H_1)}<1$ (see [28], [33]).

It is evident that, for the Wald's test, the errors of type I and II are

$$\alpha^W=p(B(\mathbf{x})>A\,|\,H_1), \tag{12}$$

and

$$\beta^W=p(B(\mathbf{x})<B\,|\,H_0). \tag{13}$$

Similar characteristics of the sequential Bayesian-type methods are:

$$\alpha^B = p(B(\mathbf{x}) > A' \mid H_1), \quad (14)$$

and

$$\beta^B = p(B(\mathbf{x}) < B' \mid H_0), \quad (15)$$

where

$$A' = \lambda \frac{p(H_0)}{p(H_1)} \text{ and } B' = \frac{1}{\lambda} \frac{p(H_0)}{p(H_1)}, \quad (16)$$

or

$$A' = \frac{1}{\lambda} \frac{p(H_0)}{p(H_1)} \text{ and } B' = \lambda \frac{p(H_0)}{p(H_1)},$$

depending on the value of $\lambda \frac{p(H_0)}{p(H_1)}$.

It is obvious that, in the general case, these characteristics for considered methods are different. Let us investigate the ratio among these probabilities. In particular, let us show in which conditions the inequalities

$$\alpha^B < \alpha^W \text{ and } \beta^B < \beta^W \quad (17)$$

are fulfilled.

It is clear that at $\lambda \frac{p(H_0)}{p(H_1)} > 1$ when

$$\lambda > \frac{p(H_1)}{p(H_0)} \frac{1-\beta}{\alpha} \text{ and } \frac{1}{\lambda} < \frac{p(H_1)}{p(H_0)} \frac{\beta}{1-\alpha}, \quad (18)$$

conditions (17) are fulfilled; at $\lambda < 1$ when

$$\frac{1}{\lambda} > \frac{p(H_1)}{p(H_0)} \frac{1-\beta}{\alpha} \text{ and } \lambda < \frac{p(H_1)}{p(H_0)} \frac{\beta}{1-\alpha}. \quad (19)$$

conditions (17) are fulfilled.

It was proved in [27] that for the given level in (6) at increasing divergence between the tested hypotheses, coefficient λ in (16) decreases and, in the limit, tends to zero. It was proved also that for the given level in (6) at decreasing divergence between the tested hypotheses coefficient λ tends to the constant which is determined by a priori probabilities of tested hypotheses, and, when these probabilities are identical, it is equal to the number of tested hypotheses minus one.

Hence it follows that there always exists such a positive value of the divergence between the hypotheses that, if the divergence between the tested hypotheses is more than that value, the method of sequential analysis of the Bayesian type rigorously surpasses the criterion with the errors of the first and the second kinds equal to α and β, respectively.

Let us suppose that $p(H_0) = p(H_1) = 0.5$, $\alpha = 0.05$, $\beta = 0.05$. From the first and the second conditions of (4) follows that $A = 171$ and $B = 2.11$, respectively. Then from (18) it follows that, if the divergence between the tested hypotheses is such that the appropriate λ in sequential Bayesian type method is greater than 19, i.e. $\lambda > 19$ then there takes place (17).

Let us now $p(H_0) = 0.1$, $p(H_1) = 0.9$, $\alpha = 0.05$, $\beta = 0.05$. Then from (18) follows that if divergence between tested hypotheses is such that the appropriate λ in the sequential Bayesian- type method is greater than 2.11, $\beta^B < \beta^W$ takes place, and, if $\lambda > 171$ both inequalities in (4) are fulfilled.

Let us consider $p(H_0) = 0.9$, $p(H_1) = 0.1$, $\alpha = 0.05$, $\beta = 0.05$. Then from (18) it follows that if the divergence between the tested hypotheses is such that the appropriate λ in the sequential Bayesian-type method is greater than 2.11, $\alpha^B < \alpha^W$ takes place, and, if $\lambda > 171$ both inequalities in (4) are fulfilled.

The results of given elementary computations correspond completely to the logical judgment. Particularly, the smaller is the a priori probability of the hypothesis, the higher is the probability of its incorrect rejection (error of type I). Analogously, the bigger is the a priori probability of the hypothesis the higher is the probability of its incorrect acceptance (error of type II).

Let us present the computation results of some examples to confirm in practice the abovementioned considerations.

Example 1. Tested hypotheses: $H_1 : \theta_1^1 = 1, \theta_2^1 = 1$; $H_2 : \theta_1^2 = 4, \theta_2^2 = 4$. A priori probabilities of the hypotheses: $p(H_1) = 0.5$, $p(H_2) = 0.5$. The significance level of the criterion in the constrained Bayesian task is $\alpha = 0.05$. The above-considered tests were applied to the sequentially incoming observation results generated as two-dimensional normally distributed random vectors with the mathematical expectation $\boldsymbol{\theta} = (4;4)$ and the covariance matrices $\mathbf{W} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{W} = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$, $\mathbf{W} = \begin{pmatrix} 10 & 5 \\ 5 & 10 \end{pmatrix}$, $\mathbf{W} = \begin{pmatrix} 20 & 15 \\ 15 & 20 \end{pmatrix}$, $\mathbf{W} = \begin{pmatrix} 30 & 25 \\ 25 & 30 \end{pmatrix}$.

This means that five samples of normally distributed random vectors with different covariance matrices were processed by both tests.

The Kullback's divergence [34] between the tested hypotheses for different samples changes depending on covariance matrices and are equal to 0.5721, 0.7171, 1.0954, 1.6036 and 4.2426, respectively. In Fig. 1 the dependences of the averaged numbers of observations necessary for making the decision in the Wald's and Bayes-type sequential tests depending on the divergences between the hypotheses are given. In Fig. 2 the dependences of the type I and II errors probabilities on the divergence for the numbers of observations equal to the averaged values for which decisions are made in the Bayes-type sequential test are presented. In Fig. 2, for each divergence the appropriate values of the number of averaged observations are shown. The same values, in the suitable sequence, for the Wald's test are 19.6, 15.6, 8.2, 2.71 and 1.66, respectively. Form here it is seen that the Bayes-type sequential test needs in average less number of observations than the Wald's test for making the decision for the considered example. The discrepancy between the averaged values is bigger the smaller is divergence. Though, the type II error probability for the Bayes-type sequential test is bigger than the analogous characteristic for the Wald's test. The Bayes-type sequential test becomes more powerful than the Wald's test by both types of errors for the hypotheses with the divergence greater than 4 (see Fig. 2).

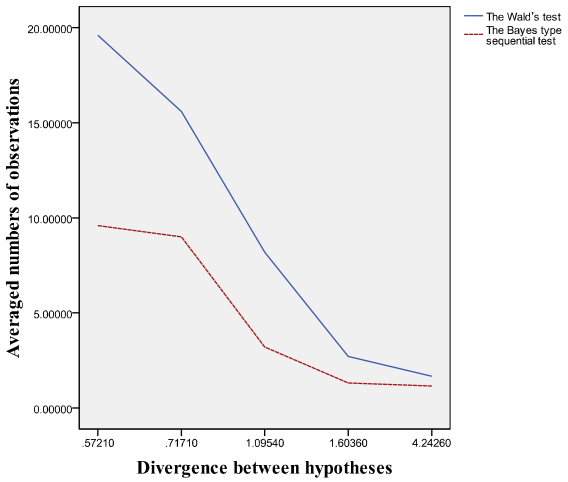

Fig. 1 The dependences of the averaged numbers of observations necessary for making decision in the Wald's and the Bayes type sequential tests depending on the divergences between hypotheses

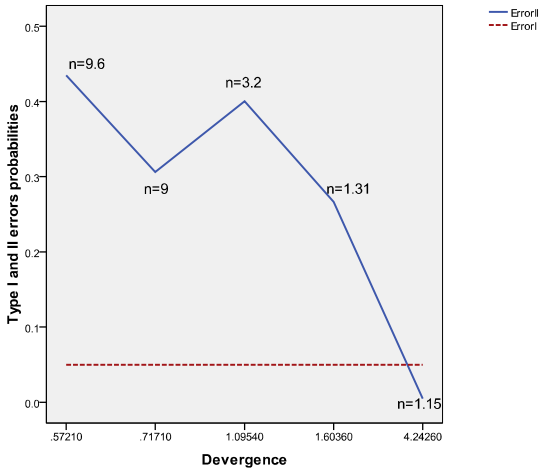

Fig. 2 The dependences of the type I and II errors probabilities on the divergence for the numbers of observations equal to the averaged values for which decisions are made in the Bayes type sequential test

V. Conclusion

On the basis of above-given results, we can conclude that the new sequential Bayes-type method is a good tool for testing any number of hypotheses. The method is universal, convenient and reliable for testing any number of hypotheses without additional investigations. The working properties of the test and the quality of the obtained results by this test are investigated by their comparison with the Wald's test for two multivariate hypotheses. The comparison allowed us to infer that, for making the decision, the new method needs in average a smaller number of observations and, moreover, for some values of divergence between the tested hypotheses, it gives more powerful decisions than the Wald's test does. It is beyond question that, in contradistinction to the Wald's test, this test is optimum, convenient and simply defined for any number of hypotheses.

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