

Numerical Solution of a Laminar Viscous Flow Boundary Layer Equation Using Uniform Haar Wavelet Quasi-linearization Method

Harpreet Kaur, Vinod Mishra, and R. C. Mittal

Abstract—In this paper, we have proposed a Haar wavelet quasi-linearization method to solve the well known Blasius equation. The method is based on the uniform Haar wavelet operational matrix defined over the interval $[0, 1]$. In this method, we have proposed the transformation for converting the problem on a fixed computational domain. The Blasius equation arises in the various boundary layer problems of hydrodynamics and in fluid mechanics of laminar viscous flows. Quasi-linearization is iterative process but our proposed technique gives excellent numerical results with quasi-linearization for solving nonlinear differential equations without any iteration on selecting collocation points by Haar wavelets. We have solved Blasius equation for $1 \leq \alpha \leq 2$ and the numerical results are compared with the available results in literature. Finally, we conclude that proposed method is a promising tool for solving the well known nonlinear Blasius equation.

Keywords—Boundary layer Blasius equation, collocation points, quasi-linearization process, uniform haar wavelets.

I. INTRODUCTION

THE solutions of the one-dimensional third order boundary value problem described by the well known blasius equation is similarity solution of the two dimensional incompressible laminar boundary layer equations. Tsou et al. [1] made a numerically and theoretically experiment on this problem to prove that a blasius flow is physically realizable. A recent study by Boyd [2], [3] pointed out how this particular problem of boundary layer theory has arisen the interest of promient scientist. In fluid mechanics, the problems are usually governed by systems of partial differential equations. If somehow, a system can be reduced to a single ordinary differential equation, this constitutes a considerable mathematical simplification of the problem. If the number of independent variables can be reduced, then partial differential equations can be replaced by ordinary differential equation. In the modeling of boundary layer, this is sometimes possible and in some cases, the system of partial differential equations reduces to a system involving a third order differential equation.

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Unfortunately, since Blasius equation is non-linear, there is not known analytic solution in closed form. The Blasius problem models the behavior of two-dimensional steady state laminar viscous flow of an incompressible fluid over a semi-infinite flate plate, provided the boundary layer assumptions are verified is governed by the continuity and the Navier-Stokes equations of motion for classical Blasius flate plate flow prob [4] and governing equations are simplified to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = \mu \frac{\partial^2 u}{\partial^2 y} \quad (2)$$

$$u(x, 0) = v(x, 0) = 0 \text{ or } u = v = 0 \text{ at } x = 0 \text{ or}$$

where $y = 0, U = U_\infty$ at $x = 0$.

The boundary conditions for this case are that both components of the velocity are zero at the wall due to no slip, and that the horizontal velocity approaches the constant free stream velocity at some distance away from the plate.

Assuming that the leading edge of the plate is x_0 and the plate is infinity long. To make this quantity dimensionless, it can be divided by y to obtain where f_u the dimensionless stream function is. The velocity component u can be expressed as follows: $u(x, y) \rightarrow \infty$ as $y \rightarrow \infty$ for the Blasius flate plate flow introducing a similarity variable and a dimensionless stream function $f(\eta)$ as;

$$\eta = y \sqrt{\frac{U}{\nu x}} = \frac{y}{x} \sqrt{\text{Re}_x} \quad (3)$$

$$\frac{u}{U} = f' ; v = \frac{1}{2} \sqrt{\frac{U \nu}{x}} (\eta f' - f) \quad (4)$$

where Re_x is the local Reynolds number $\left(= \frac{Ux}{\nu} \right)$. We obtain by applying (3) and (4).

$$\frac{\partial u}{\partial x} = -\frac{U}{2} \frac{\eta}{x^2} f'' ; \frac{\partial v}{\partial y} = \frac{U}{2} \frac{\eta}{x} f'' \quad (5)$$

And the equation of continuity is satisfied identically on the other hand, we get

$$\frac{\partial u}{\partial y} = U f'' \sqrt{\frac{U}{\nu x}}; \quad \frac{\partial^2 u}{\partial y^2} = \frac{U^2}{\nu x} f''' \quad (6)$$

Note that in (3)-(6), $U = U_\infty$ represents Blasius whereas $U = uw$ indicates sakiandis flow, respectively. By inserting (4)-(6) in (2), this system can be simplified further to an ordinary differential equation. To do this, we have an equation that reads

$$\alpha f'''(\eta) + f(\eta)f''(\eta) = 0 \quad (7)$$

Transformed boundary conditions for the momentum (7) are $f = f' = 0$ at $\eta = 0$ and $f' \rightarrow 1$ as $\eta \rightarrow \infty$.

In Blasius equation, the second derivative of $f(\eta)$ at zero plays an important role. Howarth [5] solved the Blasius equation numerically and found $f''(0) = 0.33206$. Asaithambi [6] solved the Blasius equation more accurately and obtained this number as $f''(0) = 0.332057336$. Some researchers have solved the problem numerically and some analytically. However, the solutions obtained were not very accurate. A homotopy perturbation solution to this problem was presented by Fang [7], He et al. [8] Ahmad [9], [10] also obtained the solution of Blasius problem using approximate analytical method. Liao [11], [12] obtained an analytic solution for the Blasius equation which is valid in the whole region of the problem. He constructed a five-term approximate-analytic solution for the Blasius using the variational iteration method [13] and Abbasbandy [14] obtained numerical solution of Blasius equation by adomians decomposition method. Many calculations should be done to construct the resulting semi-analytic solutions and this increases considerably the CPU time especially when a large number of terms of solutions are to be used. From the review of the proposed schemes, two general limitations may be observed: The proposed approximate-analytic methods cannot yield accurate solutions when a rather small number of solution terms are used.

Our main goal here is to show how to solve numerically the blasius problem by Haar wavelet approximation. In this work, the Haar wavelet quasilinearization (HWQ) process is proposed to solve the classical Blasius flat-plate problem. The numerical results are obtained via proposed method for $\alpha = 1, 1.2, 1.5, 1.8$ and 2 and compared with the available results in literature.

II. STRUCTURE OF HAAR WAVELETS BASED ON MULTI RESOLUTION ANALYSIS (MRA)

Wavelets were ripe for discovery in the 1980s. The great impetus came from two discoveries: the multiresolution of Mallat or Meyer and most of all the discovering by Daubechies [15] compactly supported orthogonal wavelets

with arbitrary smoothness. Wavelets generalize readily to several dimensions.

The Haar wavelet function was introduced by Alfred Haar in 1910 [16] in the form of a rectangular pulse pair function. After that many other wavelet functions were generated and introduced. Those include the Shannon, Daubechies [15] and Legendre wavelets. Among these forms, Haar wavelet is the only real valued wavelet that is compactly supported, symmetric and orthogonal. The basic and simplest form of Haar wavelet is the Haar scaling function that appears in the form of a square wave over the interval $t \in [0, 1]$, generally written as;

$$h_1(t) = \begin{cases} 1 & t \in [0, 1) \\ 0 & \text{elsewhere} \end{cases} \quad (8)$$

The above expression, called Haar father wavelet, is the zeroth level wavelet which has no displacement and dilation of unit magnitude. The following definitions illustrate the translation dilation of wavelet function.

There are many excellent accounts of multiresolution and wavelet theory.

The sequence $\{\psi_i\}_{i=0}^{\infty}$ is a complete orthonormal system in $L^2[0, 1]$ and by using the concept of multiresolution analysis (MRA) as an example the space V_j can be defined like

$$\begin{aligned} V_j &= sp\{\psi_{j,k}\}_{k=0,1,2,\dots,2^j-1} \\ &= W_{j-1} \oplus V_{j-1} = W_{j-1} \oplus W_{j-2} \oplus V_{j-2} \oplus \dots = \bigoplus_{i=0}^{j-1} W_i \oplus V_0, \end{aligned} \quad (9)$$

The linearly independent functions $\psi_{j,k}(t)$ spanning W_j are called wavelets. Original signal can be expressed as a linear combination of the box basis functions in V_j . The functions $\psi(t)$ and $\psi_{j,k}(t)$ are all orthogonal in $[0, 1]$, with

$$\int_0^1 \psi(t)\psi_{j,k}(t)dt = 0 \quad \text{and} \quad \int_0^1 \psi_{j,k}(t)\psi_{l,m}(t)dt = 0 \quad (10)$$

For $(j, k) \neq (0, 0)$ in the first case and $(j, k) \neq (l, m)$ in the second.

The Haar mother wavelet is the first level Haar wavelet and can be written as the linear combination of the Haar scaling function by using

$$h_2(t) = h_1(2t) + h_1(2t-1) \quad (11)$$

Similarly, the other wavelets can be generated with two operations of translation and dilation. Each Haar wavelet is composed of a couple of constant steps of opposite sign during its subinterval and is zero elsewhere. The term wavelet is used

to refer to a set of orthonormal basis functions generated by dilation and translation of a compactly supported scaling function $h_1(t)$ (father wavelet) and a mother wavelet $h_2(t)$ associated with an multiresolution analysis of $L^2(R)$. Thus we can write out the Haar wavelet family as

$$h(t) = h(2^j t - k) = \begin{cases} 1 & \frac{k}{2^j} \leq t < \frac{k+0.5}{2^j} \\ -1 & \frac{k+0.5}{2^j} \leq t \leq \frac{k+1}{2^j} \\ 0 & \text{elsewhere} \end{cases} \quad (12)$$

For $i \geq 2$, $i = 2^j + k + 1$, $j \geq 0$, $0 \leq k \leq 2^j - 1$ and collocation points are defined as $t_l = \frac{l-1}{2m}$, $l = 1, 2, \dots, 2m$.

Here m is the level of the wavelet, we assume the maximum level of resolution is index J , then $m = 2^j$, ($j = 0, 1, 2, \dots, J$); in case of minimal values $m = 1, k = 0$ then $i = 2$. For any fixed level m , there are m series of i to fill the interval $[0, 1)$ corresponding to that level and for a provided J , the index number i can reach the maximum value $M = 2^{j+1}$, when including all levels of wavelets.

We can find the required derivatives in terms of operational matrix. The operational matrix $p_{i,n}(t)$ of order $2m \times 2m$ can be obtained by integration of Haar wavelet. Integrals can be evaluated from (12) and the first two of them are given below. Also for the ease of implementation, we have used the same notations for Haar wavelets and their integrals as [17].

$$p_{i,2}(t) = \begin{cases} \frac{1}{2} \left(t - \frac{k}{m} \right)^2 & t \in \left[\frac{k}{m}, \frac{k+0.5}{m} \right) \\ \frac{1}{4m^2} - \frac{1}{2} \left(\frac{k+1}{m} - t \right)^2 & t \in \left[\frac{k+0.5}{m}, \frac{k+1}{m} \right) \\ \frac{1}{4m^2} & t \in \left[\frac{k+1}{m}, 1 \right) \\ 0 & \text{elsewhere} \end{cases} \quad (13)$$

III. TRANSFORMATION OF BLASIUS EQUATION AND SOLUTION PROCEDURE TO SOLVE THE PROBLEM

We begin now the development of the numerical procedure for solving the Blasius problem. The transformation $\eta = \tan\left(\frac{\pi t}{2}\right)$ and a collocation method with orthogonal Haar wavelets are introduced to solve numerically the third order nonlinear Blasius differential (7).

Under the transformation $\eta = \tan\left(\frac{\pi t}{2}\right)$, derivatives are derived as

$$\frac{d^2 f}{d\eta^2} = \left(\frac{d^2 f}{dt^2} \cos^2(\pi t / 2) + \frac{df}{dt} \left(-2 \cos(\pi t / 2) \sin(\pi t / 2) \cdot \frac{\pi}{2} \right) \right) \frac{4 \cos^2(\pi t / 2)}{\pi^2} \quad (14)$$

$$\frac{d^3 f}{d\eta^3} = \frac{8}{\pi^3} \frac{d^3 f}{dx^3} \cos^6(\pi t / 2) - \frac{4(4 \cos(\pi t / 2) + 1)}{\pi^2} \cos^4(\pi t / 2) \sin(\pi t) \frac{df}{dt} + \frac{2}{\pi} \cos^2(\pi t / 2) \frac{df}{dt} (-2 \cos^2(\pi t / 2) \cos(\pi t) + \sin^2(\pi t)) \quad (15)$$

The proposed technique is based on operational matrix at collocation points. The operational matrix is derived from integration of Haar wavelet family by Chen and Hsiao in 1997 [17]. The Haar basis has the very important property of multiresolution analysis that $V_{j+1} = V_j \oplus W_j$. The orthogonality property puts a strong limitation on the construction of wavelets and allows us to transform any square integral function on the interval time $[0, 1)$ into Haar wavelets series as

$$f(t) = c_0 h_0(t) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} c_{2^j+k} h_{2^j+k}(t), \quad t \in [0, 1] \quad (16)$$

Similarly the highest derivative can be written as wavelet series $\sum_{i=-\infty}^{\infty} a_i h_i(t)$. In applications, Haar series are always truncated to $2m$ terms, that is $\sum_{i=0}^{2m} a_i h_i(t)$ [17], then we have

used the quasi-linearization process. The quasi-linearization process is an application of the Newton Raphson Kantorovich approximation method in function space [18]. The idea and advantage of the method is based on the fact that linear equations can often be solved analytically or numerically while there are no useful techniques for obtaining the general solution of a nonlinear equation in terms of a finite set of particular solutions. Consider an n^{th} order nonlinear ordinary differential equation

$$L^{(n)} f(t) = g(f(t), f^{(1)}(t), f^{(2)}(t), f^{(3)}(t), \dots, f^{(n-1)}(t), t) \quad (17)$$

with the initial conditions

$$f(0) = \lambda_0, f^{(1)}(0) = \lambda_1, f^{(2)}(0) = \lambda_2, \dots, f^{(n-1)}(0) = \lambda_{n-1} \quad (18)$$

Here $L^{(n)}$ is the linear n^{th} order ordinary differential operator, g is nonlinear function of $f(t)$ and its $(n-1)$ derivatives are $f^{(s)}(t)$, $s = 0, 1, 2, \dots, n-1$.

The quasi-linearization prescription determines the $(r+1)^{\text{th}}$ iterative approximation to the solution of (17) and its linearized form is given by (19).

$$L^{(n)} f_{r+1}(t) = g(f_r(t), f_r^{(1)}(t), f_r^{(2)}(t), f_r^{(3)}(t), \dots, f_r^{(n-1)}(t), t) + \sum_{s=0}^{n-1} (f_{r+1}^{(s)}(t) - f_r^{(s)}(t)) g_{u^{(s)}}(f_r(t), f_r^{(1)}(t), f_r^{(2)}(t), \dots, f_r^{(n-1)}(t), t) \quad (19)$$

where $f_r^{(0)}(t) = f_r(t)$. The functions $g_{u^{(s)}} = \frac{\partial g}{\partial f^s}$ are functional derivatives of the functions. The zeroth approximation $f_0(t)$ is chosen from mathematical or physical considerations.

We linearize the nonlinear (7) by using quasi-linearization process and followed by simplification yields

$$f(t)f''(t) = f_r(t)f_r''(t) + (f_{r+1}(t) - 2f_r(t))f_r''(t) + f_{r+1}''(t)f_r(t) \quad (20)$$

Then by following Haar wavelet quasilinearization method [19]-[21], equation can easily be written as the system.

$$\left[\begin{aligned} &\alpha \frac{16}{\pi^3} \cos^6(\pi t_i) \sum_{i=0}^{2m} a_i h_i + \sum_{i=0}^{2m} a_i p_{i,3} \\ &\left(\frac{4}{\pi^2} \cos^4(\pi t_i / 2) - \frac{2t_i}{\pi} \cos^2(\pi t_i / 2) \sin(\pi t_i) \right) \end{aligned} \right] \\ + \sum_{i=0}^{2m} a_i p_{i,2} \left(\begin{aligned} &\frac{4}{\pi^2} \cos^4(\pi t_i / 2) (-2 \cos^2(\pi t_i / 2) \cos(\pi t_i) + \sin^2(\pi t_i)) \\ &-\frac{1}{\pi} t_i^2 \cos^2(\pi t_i / 2) \sin(\pi t_i) \end{aligned} \right) \\ + \sum_{i=0}^{2m} a_i p_{i,1} \left(\begin{aligned} &\frac{-8}{\pi^2} \cos^4(\pi t_i / 2) (2 \cos(\pi t_i) \sin(\pi t_i / 2) + \sin(\pi t_i)) + \\ &\frac{2t_i^2}{\pi^2} \cos^4(\pi t_i / 2) \end{aligned} \right) + \\ \sum_{i=0}^{2m} a_i p_{i,21} \left(\begin{aligned} &\frac{8}{\pi^2} \cos^4(\pi t_i / 2) (2 \cos(\pi t_i) \sin(\pi t_i / 2) + \sin(\pi t_i)) \\ &-\frac{4t_i}{\pi^2} \cos^2(\pi t_i / 2) (-2 \cos^2(\pi t_i / 2) \cos(\pi t_i) + \sin^2(\pi t_i)) \\ &-\frac{4t_i^2}{\pi^2} \cos^4(\pi t_i / 2) - \frac{2t_i^3}{\pi} \cos^2(\pi t_i / 2) \sin(\pi t_i) \end{aligned} \right) \\ = \frac{8}{\pi^2} \cos^4(\pi t_i / 2) \sin(\pi t_i) (2 \cos(\pi t_i) + 1) - \frac{4t_i}{\pi^2} \cos^2(\pi t_i / 2) \\ (-2 \cos^2(\pi t_i / 2) \cos(\pi t_i) + \sin^2(\pi t_i)) - \frac{2t_i^2}{\pi^2} \cos^4(\pi t_i / 2) \quad (21)$$

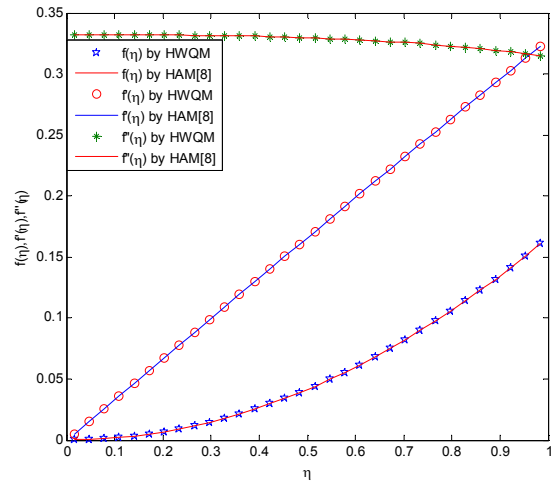


Fig. 1 Plot of Comparison of Present Method for $\alpha = 2$

IV. NUMERICAL RESULTS AND DISCUSSION

In Blasius equation, the second derivative of $f(\eta)$ at zero plays an important role. Numerical solutions by applying the proposed technique to several values of η and $\alpha = 2$ for $f(\eta)$ and its derivatives are given in Table I and comparison is shown in Fig. 1. For different values of $\alpha = 1, 1.2, 1.5, 1.8$ $f''(\eta)$ are computed and depicted in Table II and Fig. 2.

Further the quantity $\frac{d^2 f}{d\eta^2}$ for $\eta = 0$ has been computed i.e $f''(0) = 332057$. Computational work is computed by C++ and MATLAB R2007b for wavelet mode $m = 32$.

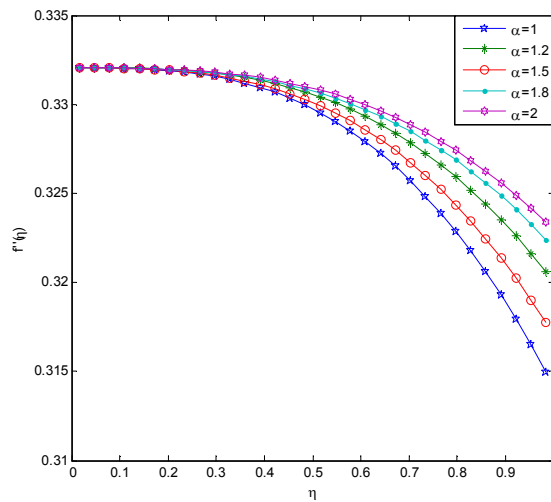


Fig. 2 Plot of $f''(\eta)$ to the Blasius Flow for Different Values of α

TABLE I
WAVELET SOLUTION $f(\eta)$ AND ITS DERIVATIVES FOR DIFFERENT VALUES OF η AND $\alpha = 2$

η	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
0.0156	4.053e-005	0.005188	0.332057
0.0469	0.0003648	0.015565	0.332054
0.0781	0.0010133	0.025941	0.332046
0.1094	0.0019861	0.036318	0.332037
0.1719	0.0049046	0.057067	0.331954
0.2031	0.0068503	0.067440	0.331893
0.2344	0.0091202	0.077810	0.331805
0.2656	0.0117144	0.088178	0.331698
0.2969	0.0146329	0.098541	0.331563
0.3281	0.0178756	0.108901	0.331396
0.3594	0.0214426	0.119254	0.331193
0.3906	0.0253339	0.129643	0.330951
0.4219	0.0295495	0.139938	0.330666
0.4531	0.0340893	0.150266	0.330337
0.4844	0.0389535	0.160583	0.329959
0.5156	0.0441419	0.170888	0.329533
0.5469	0.0496545	0.181179	0.329049
0.5781	0.0554915	0.191453	0.328516
0.6094	0.0616527	0.201712	0.327911
0.6406	0.0681382	0.211947	0.327246
0.6719	0.0749479	0.222162	0.326515
0.7031	0.0820826	0.232354	0.325713
0.7344	0.0895403	0.242519	0.324838
0.7969	0.1054305	0.262761	0.322865
0.8281	0.1138610	0.272834	0.321759
0.8594	0.1226164	0.282871	0.320571
0.8906	0.1316967	0.292869	0.319299
0.9531	0.1508286	0.312745	0.316491
0.9844	0.1608813	0.322606	0.314951

V.CONCLUSION

In this work Haar wavelet method is applied to solve nonlinear Blasius equation. To the best of our knowledge, the method of quasi-linearization has not been used for above nonlinear problem with Haar wavelets. The advantage of quasi-linearization is that one does not have to apply iterative procedure. The results of the comparison with other method indicate that the proposed method is feasible. Also the effect of constant parameters on response of system for Haar wavelet method is also shown by figures. It is also shown that the use of the quasi-linearization process and proposed transformation makes easier by Haar wavelet method to handle nonlinearity in a shorter time of computations. We observed that $f''(\eta)$ at any point near the $\eta = 0$ decreases when α increases.

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TABLE II
COMPUTED $f''(\eta)$ FOR DIFFERENT VALUES OF $\alpha = 1, 1.2, 1.5, 1.8$

η	$f''(\eta)$		
	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 1.8$
0.0156	0.332057	0.332057	0.332055
0.0469	0.332054	0.332057	0.332052
0.0781	0.332046	0.332055	0.332048
0.1094	0.332034	0.332049	0.332038
0.1406	0.332001	0.332034	0.332021
0.1719	0.331954	0.332005	0.331996
0.2344	0.331805	0.331913	0.331910
0.2656	0.331698	0.331841	0.331848
0.2969	0.331563	0.331751	0.331769
0.3594	0.331193	0.331505	0.331557
0.3906	0.330951	0.331346	0.331421
0.4219	0.330666	0.331159	0.331262
0.4531	0.330337	0.330941	0.331077
0.4844	0.329959	0.330692	0.330868
0.5156	0.329532	0.330405	0.330629
0.5781	0.328513	0.329723	0.330052
0.6094	0.327915	0.329319	0.329712
0.6406	0.327246	0.328873	0.329337
0.6719	0.326515	0.328382	0.328927
0.7344	0.324838	0.327262	0.327988
0.7656	0.323891	0.326627	0.327455
0.7969	0.322865	0.325941	0.326889
0.8281	0.321759	0.325196	0.326260
0.8594	0.320571	0.324396	0.325591
0.9219	0.317943	0.322620	0.324104
0.9531	0.316491	0.321638	0.323284
0.9844	0.314951	0.320595	0.322409

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