Exterior Calculus: Economic Profit Dynamics

Troy L. Story

***Abstract*—**A mathematical model for the Dynamics of Economic Profit is constructed by proposing a characteristic differential one-form for this dynamics (analogous to the action in Hamiltonian dynamics). After processing this form with exterior calculus, a pair of characteristic differential equations is generated and solved for the rate of change of profit *P* as a function of revenue *R* (*t*) and cost *C* (*t*). By contracting the characteristic differential one-form with a vortex vector, the Lagrangian is obtained for the Dynamics of Economic Profit.

Keywords—Differential geometry, exterior calculus, Hamiltonian geometry, mathematical economics, economic functions, and dynamics

I. Introduction

USING as a background a recent paper on the application of exterior calculus to Economic Growth Dynamics [1], the present paper uses exterior calculus to synthesize a set of differential equations for the dynamics of Economic Profit *P*(*R* , *C, t*), where *R*(*t*) is the revenue, *C*(*t*) is the cost and *t* is the time. The differential equations produced are solved for the rate of change of price as a function of the revenue and cost. The following principle is used:

Mathematical models of dynamics employing exterior calculus are mathematical representations of the same unifying principle; namely, the description of a dynamic system with a characteristic differential one-form on an odd-dimensional differentiable manifold leads, by analysis with exterior calculus, to a set of differential equations and a characteristic tangent vector which define transformations of the system [2]. The origin of this principle is Arnold's [3] use of differential forms to define Hamiltonian geometry.

The background for the mathematical structure of the present mathematical model has been presented before [1,2,4, 5]; however, for the convenience of the reader, section II contains a discussion of differential forms, and then in section III, it is shown how differential one-forms are used to develop a model for a dynamic system. With this preparation, the model for dynamics on differential forms is applied to Economic Profit dynamics in section IV.

The model allows computation of the rate of change of the profit *P* (*R* , *C, t*) as a function of the revenue *R* (*t*) and the cost *C*(*t*); these results are entirely dependent on the use of this differential geometric approach.

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II. Differential One-Forms

The exterior derivative of a scalar function f in exterior calculus (a differential one-form $\mathbf{d}f$) is the same operation on f as the exact differential of a scalar function f in conventional calculus (exact differential df); namely, the operation represents an infinitesimal change in f induced by an arbitrary displacement of a point. However, df is already a scalar, whereas $\mathbf{d}f$ must be contracted with a tangent vector $\mathbf{v}$ to become a scalar. The operation of contraction, denoted by $\mathbf{d}f\left(\mathbf{v}\right)$, thus removes the arbitrariness in the direction of the displacement, where this direction is the same as that of the tangent vector $\mathbf{v}$ (tangent vectors and the exterior derivative operator are denoted by boldface symbols and a boldface **d,** respectively).

In this setting, consider an *n*-dimensional differentiable manifold *M* with *n* local coordinates x^k . At every point of M,

(a) there exists a basis set of tangent vectors $\left\{\partial/\partial x^k\right\}$ for an *n*-dimensional vector space of tangent vectors $\mathbf{v}$ belonging to tangent space TM_x and

(b) there exists a basis set of differential one-forms $\left\{\mathbf{d}x^k\right\}$ for an *n*-dimensional vector space of differential one-forms $\mathbf{d}f$ on tangent space TM_x.

The tangent bundle $TM\left(=\cup TM_x\right)$ and cotangent bundle $T^*M\left(=\cup T^*M_x\right)$, where the cotangent space T^*M_x is the dual of tangent space TM_x, have the natural structure of a differential manifold of dimension $2n$ with local coordinates $\left\{x^k,\mathbf{d}x^k\left(\mathbf{v}\right)\right\}$ and $\left\{x^k,\mathbf{d}\,f\left(\partial/\partial x^k\right)\right\}$, respectively. A differential one-form $\mathbf{d}S$ on T^*M_x is defined by the contraction $\mathbf{d}S\left(\xi\right)=\mathbf{d}f\left(\mathbf{v}\right)$, where $\xi\in T\left(T^*M_x\right)$; hence,

$$\mathbf{d}S=\mathbf{d}\,f\left(\partial/\partial x^k\right)\mathbf{d}x^k \tag{1}$$

III. Dynamics

In Arnold's treatment [3] of Hamiltonian mechanics and in the present case of economic profit as a dynamic system, a temporal coordinate x^0 is introduced as an additional local coordinate for M, TM and T^*M , thereby changing TM and T^*M into odd-dimensional manifolds. As a result, an additional term $\mathbf{d}\,f\left(\partial/\partial x^0\right)\mathbf{d}x^0$ is added to (1), where

$\mathbf{d}\, f\left(\partial / \partial x^0\right)$ is defined as a function of all (2*n* +1) coordinates; hence, $\mathbf{d}\, f\left(\partial / \partial x^0\right)$ describes the phase flow on the extended cotangent bundle. Using b_k for $\mathbf{d}\, f\left(\partial / \partial x^k\right)$ and $\Omega\,\mathbf{d}x^0$ for $\mathbf{d}\, f\left(\partial / \partial x^0\right)\mathbf{d}x^0$, the equation for $\mathbf{d}S$ becomes

$$\mathbf{d}S = b_k \mathbf{d}x^k + \Omega\left(x^0, ...x^n, b_1, ..., b_n\right)\mathbf{d}x^0 \tag{2}$$

In Hamiltonian mechanics b_k, Ω and x^0 are represented by the momenta, Hamiltonian and time, respectively, but for the example discussed in section IV, other variables will play the role of b_k, Ω and x^0, as well as of S and x^k. Hence, for the remainder of this section the geometry of extended phase space is presented in a general setting that not only applies to Hamiltonian mechanics (which defines this geometry), geometric optics, irreversible thermodynamics, black hole dynamics, Navier-Stokes dynamics, and economic growth dynamics, but also to Economic Profit dynamics.

The general procedure begins by taking the exterior derivative of $\mathbf{d}S$ to get the following differential two-form:

$$\mathbf{d}\omega = \mathbf{d}b_k \wedge \mathbf{d}x^k + \left[\left(\frac{\partial\Omega}{\partial x^k}\right)\mathbf{d}x^k + \left(\frac{\partial\Omega}{\partial b_k}\right)\mathbf{d}b_k + \left(\frac{\partial\Omega}{\partial t}\right)\mathbf{d}t\right]\wedge \mathbf{d}t \tag{3}$$

where $\omega \equiv \mathbf{d}S$. If x^k and b_k are to describe mappings of the temporal coordinate x^0 onto the direction of the system phase flow, then (a) x^k and b_k must be functions of x^0 alone and (b) the following contraction must be satisfied at each point $\left(b_k, x^k, x^0\right)$ of the transformation:

$$\mathbf{d}\omega\left(\xi, \eta\right) = 0 \tag{4}$$

where the tangent vector ξ is given by

$$\xi = \left(\frac{db_k}{dx^0}\right)\frac{\partial}{\partial b_k} + \left(\frac{dx^k}{dx^0}\right)\frac{\partial}{\partial x^k} + \frac{\partial}{\partial x^0} \tag{5}$$

and where η is an arbitrary vector. $\mathbf{d}\omega$ is a mapping of a pair of vectors onto an oriented surface; if the contraction $\mathbf{d}\omega\left(\xi, \eta\right) = 0$, then the mapping is defined only if the coordinates db_k / dx^0 and dx^k / dx^0 of ξ have the values

$$dx^k / dx^0 = -\left(\partial\Omega / \partial b_k\right) \quad \text{and} \quad db_k / dx^0 = \left(\partial\Omega / \partial x^k\right) \tag{6}$$

By substituting the coordinate values from (6) into (5), the vortex vector $\mathbf{R}$ is obtained, as given by

$$\mathbf{R} = \left(\frac{\partial\Omega}{\partial x^k}\right)\frac{\partial}{\partial b_k} - \left(\frac{\partial\Omega}{\partial b_k}\right)\frac{\partial}{\partial x^k} + \frac{\partial}{\partial x^0} \tag{7}$$

The foregoing discussion leads to the following two points: first, contraction of $\mathbf{d}S$ with the vortex vector $\mathbf{R}$, gives

$$\mathbf{d}S\left(\mathbf{R}\right) = -b_k\left(\partial\Omega / \partial b_k\right) + \Omega \tag{8}$$

where $\mathbf{d}S\left(\mathbf{R}\right)$ is the Lagrangian on extended tangent space $\left(x^k, dx^k / dx^0, x^0\right)$. Secondly, note that for (4), where the exterior derivative of a characteristic differential one-form is contracted on a pair of tangent vectors and set equal to the unique scalar zero, the analysis refers to vortex tubes which do not end. For vortex tubes which end in an elementary volume, $\mathbf{d}\omega\left(\xi, \eta\right)$ is set equal to a unique scalar other than zero. A previous application [2] of the present model to the source dependent Maxwell equations illustrates the difference in procedure required for such vortex tubes.

These results lead to the following proposal for all physical processes assumed to proceed in a characteristic direction. Mathematical models of dynamics employing exterior calculus are mathematical representations of the same unifying principle; namely, the description of a dynamic system with a characteristic differential one-form on an odd-dimensional manifold, leads by analysis with exterior calculus to a set of differential equations and a vortex vector which define transformations of the systems.

IV. Economic Profit Dyamics On A Differential One-Form

The principle described in sections II and III is now applied to Economic Profit dynamics. In analogy with Hamiltonian dynamics, the present investigation proposes a differential one-form for Economic Profit dynamics on an odd-dimensional differentiable manifold. It is then shown that the use of exterior calculus predicts a pair of differential equations and a characteristic tangent vector (the vortex vector) for this dynamics. This pair of equations are solved for the rate of change of Economic Profit with respect to the revenue *R* (*t*) and cost *C* (*t*). By contracting the characteristic differential one-form with the vortex vector, the Lagrangian is obtained for Economic Profit dynamics.

A. Differential one-form for Economic Profit dynamics; Dynamics

Using as a starting point the Omega function $P\left(R_i, C^i, t\right)$, the differential one-form proposed for Economic Profit dynamics is

$$\mathbf{d}W_P = R_i\,\mathbf{d}C^i + P\left(R_i, C^i, t\right)\mathbf{d}t \tag{9}$$

where W_P plays the role of the action in Hamiltonian mechanics, $P(R_i, C^i, t)$ is the characteristic function (the Omega function, e.g., the Hamiltonian), R_i is the revenue, C^i is the cost, and t is the time. The variable $R_i(t)$ (compare to momentum in Hamiltonian mechanics) is conjugate to the "position" variable $C^i(t)$, as indicated by the following conditions for conjugacy:

$$\begin{aligned}&(\mathrm{a})\, R_i = \mathbf{d}W_P\left(\partial/\partial C^i\right) = \text{contraction of } \mathbf{d}W_P \text{ with } \partial/\partial C^i\\&(\mathrm{b})\, R_i = R_i(t) \quad \text{and} \quad C^i = C^i(t)\\&(\mathrm{c})\, P = P(R_i, C^i, t)\end{aligned} \tag{10}$$

Using the symbol $\omega \equiv \mathbf{d}W_P$, the exterior derivative of $\mathbf{d}W_P$ is

$$\begin{aligned}\mathbf{d}\omega &= \mathbf{d}R_i \wedge \mathbf{d}C^i\\&-\left[\left(\frac{\partial P}{\partial C^i}\right)\mathbf{d}C^i + \left(\frac{\partial P}{\partial R_i}\right)\mathbf{d}R_i + \left(\frac{\partial P}{\partial t}\right)\mathbf{d}t\right] \wedge \mathbf{d}t\end{aligned} \tag{11}$$

Following the procedure of Story ([1], [2], [4], and [5]), consider the vectors $\xi, \eta \in T(T^*M_C)$, where $T(T^*M_C)$ is the tangent of the cotangent space at point C^i on the manifold and where vector ξ and arbitrary vector η are

$$\xi = \left(\frac{dR_i}{dt}\right)\frac{\partial}{\partial R_i} + \left(\frac{dC^i}{dt}\right)\frac{\partial}{\partial C^i} + \frac{\partial}{\partial t} \tag{12}$$

$$\eta = \beta_{R_i}\frac{\partial}{\partial R_i} + \beta_{C^i}\frac{\partial}{\partial C^i} + \frac{\partial}{\partial t} \tag{13}$$

Employing the mapping $\mathbf{d}\omega : (\xi, \eta) \rightarrow \mathbf{d}\omega(\xi, \eta)$, note that this mapping and the contraction

$$\mathbf{d}\omega(\xi, \eta) = 0 \tag{14}$$

are defined only it the coordinates $\frac{dC^i}{dt}$ and $\frac{dR_i}{dt}$ of ξ have the values

$$\frac{dC^i}{dt} = -\left(\frac{\partial P}{\partial R_i}\right) \quad \text{and} \quad \frac{dR_i}{dt} = +\left(\frac{\partial P}{\partial C^i}\right) \tag{15}$$

for arbitrary tangent vector η. These equations define the relationship between coordinates $\left(\frac{dR_i}{dt}, \frac{dC^i}{dt}, 1\right)$ and coordinate values $\left(\frac{\partial P}{\partial C^i}, \frac{-\partial P}{\partial R_i}, 1\right)$ for tangent vector ξ at each point of the transformation; hence, the arbitrariness in the coordinates of ξ is removed. The characteristic tangent vector obtained by replacing the coordinates for ξ from (12) with the coordinate values defined by the two differential equations (15), is called the vortex vector (section IVC). This vector gives the direction (the vortex direction) of the system phase flow, with the vortex lines (integral curves of the differential equations passing through points of a closed curve) called the vortex tube.

B. Solutions

Focusing on the differential equation $\frac{dR_i}{dt} = \left(\frac{\partial P}{\partial C^i}\right)$, note that a positive time rate of revenue earned by selling a commodity, $\left(\frac{dR_i}{dt} > 0\right)$, implies an increase in the rate of profit growth with respect to cost. The converse holds; namely, $\frac{dR_i}{dt} < 0$, implies a decrease in the rate of profit growth with respect to cost.

Focusing on the differential equation $\frac{dC^i}{dt} = -\left(\frac{\partial P}{\partial R_i}\right)$, it is noted that an increase in the speed of production costs, $\frac{dC^i}{dt} > 0$, implies a decrease in rate of profit growth with respect to revenue; conversely, a decrease in the speed of production costs, $\frac{dC^i}{dt} < 0$, implies an increase in the rate of profit growth with respect to revenue.

Consider the solutions to these characteristic differential equations. Assuming $\left(\frac{\partial P}{\partial C^i}\right)$ is constant over the time interval of interest, the equation $\frac{dR_i}{dt} = \left(\frac{\partial P}{\partial C^i}\right)$, has the solution

$$R_i = +\left(\frac{\partial P}{\partial C^i}\right)t + \text{ constant of integration} \tag{16}$$

By plotting R_i *vs* t, a straight line is predicted with a slope $\left(\frac{\partial P}{\partial C^i}\right)$; thus, the rate of change of profit with respect to costs can be computed from (R_i, t) data.

Following the same procedure for the equation $\frac{dC^i}{dt} = -\left(\frac{\partial P}{\partial R_i}\right)$, while assuming $\left(\frac{\partial P}{\partial R_i}\right)$ is constant of the time interval of interest, leads to the solution

$$C^i = -\left(\frac{\partial P}{\partial R_i}\right)t + \text{ constant of integration} \tag{17}$$

In this case the straight line predicts a slope of $-\left(\frac{\partial P}{\partial R_i}\right)$; hence, the rate of change of profit with respect to revenue can be computed from $\left(C^i,\ t\right)$ data. Solutions in eqns. (16 & 17) therefore provide a quantitative measure of the profit growth, based on observations of time rates of change of the revenue and the production costs.

C. Vortex vector, Lagrangian

By substituting the coordinate values from $\frac{dC^i}{dt} = -\left(\frac{\partial P}{\partial R_i}\right)$ and $\frac{dR_i}{dt} = +\left(\frac{\partial P}{\partial C^i}\right)$ into (12), the vortex vector is obtained as

$$\mathbf{R} = +\left(\frac{\partial P}{\partial C^i}\right)\frac{\partial}{\partial R_i} - \left(\frac{\partial P}{\partial R_i}\right)\frac{\partial}{\partial C^i} + \frac{\partial}{\partial t} \qquad (18)$$

This vector gives the direction of the system change in $\left(R_i, C^i, t\right)$-space, an extended cotangent space.

The Lagrangian of the system is obtained [2] by contracting the characteristic differential one-form $\mathbf{d}W_P$ with the vortex vector $\mathbf{R}$, giving for the Lagrangian,

$$\mathbf{d}W_P\left(\mathbf{R}\right) = -R_i\left(\frac{\partial P}{\partial R_i}\right) + P = +R_i\left(\frac{dC^i}{\partial t}\right) + P \qquad (19)$$

D. Integral Invariant of Economic Profit

Let γ_1 and γ_2 be two closed curves in a $(2n + 1)$-dimensional manifold M^{2n+1}. The vortex lines passing through points of γ_1 and γ_2 form a vortex tube for the extended phase space $\left(R_i, C^i, t\right)$ with $\gamma_1 - \gamma_2 = \partial\sigma$, where σ is a section of the vortex tube and $\partial\sigma$ is the boundary of σ. The vortex lines of $\omega\left(\equiv \mathbf{d}W_P\right)$ on the extended phase space give a one-to-one projection onto the t - axis. By Stokes' formula,

$$\oint_{\gamma_1}\omega - \oint_{\gamma_2}\omega = \int_{\partial\sigma}\omega = \int_{\sigma}\mathbf{d}\,\omega \qquad (20)$$

However, according to eqns. (15) it was shown that the equations

$$\frac{dC^i}{dt} = -\left(\frac{\partial P}{\partial R_i}\right) \quad \text{and} \quad \frac{dR_i}{dt} = +\left(\frac{\partial P}{\partial C^i}\right) \qquad (21)$$

arrive only when $\mathbf{d}\omega\left(\xi,\eta\right) = 0$. Hence, the integral of $\mathbf{d}\omega$ is zero, implying

$$\oint_{\gamma_1}\omega = \oint_{\gamma_2}\omega \qquad (22)$$

Eqn. (22) implies $\omega = \mathbf{d}W_P = R_i\,\mathbf{d}C^i + P\left(R_i, C^i, t\right)\mathbf{d}t$ is an integral invariant of Economic Profit Dynamics.

V. Conclusion

The principle applied in this paper is identical to the one applied in other areas of Hamiltonian geometry (geometric optics, thermodynamics, Black holes, classical electromagnetism, classical string theory, Navier-Stokes dynamics, and economic growth dynamics). By applying exterior calculus to economic profit dynamics, a set of differential equations and a characteristic tangent vector for economic profit are constructed. Solution of these equations gave rates of change of the profit with respect to revenue and cost. Since a critical and quantitative means of measuring economic profit as a function of revenue and cost is an extremely useful societal tool, it is expected that the results presented here will focus more attention to this area of mathematical economics and to other applications of this differential geometric model of dynamics.

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