On uniqueness and continuous dependence in the theory of micropolar thermoelastic mixtures

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Abstract—This paper studies questions of continuous data dependence and uniqueness for solutions of initial boundary value problems in linear micropolar thermoelastic mixtures. Logarithmic convexity arguments are used to establish results with no definiteness assumptions upon the internal energy.

Keywords—Cellular materials, Continuous dependence, Micropolar mixtures, Uniqueness.

I. INTRODUCTION

Motivated by the fact that many natural or synthetic materials are not pure materials, Twiss and Eringen [1], [2] have introduced the mixture theory of materials with microstructure. Sometimes the presence of a constituent can be ignored, if there is a preponderant constituent, but in many situations the local mechanical effects of each ingredient of the mixture cannot be ignored. A mixture is thereby envisioned as a superposition of several continuous media. Two papers of Eringen [3], [4] got back in discussion the study of mixtures with microstructure.

The theories of mixtures with microstructure have many applications in consolidation problems in the building industry and oil exploration problems [5]. A mathematical model intended to describe the microstructure of materials is the micropolar media [6]. For this model, each particle can independently translate and rotate, so that it has six degrees of freedom. Elangovan et al. [7] combine micropolar elasticity theory and Eringen's mixture theory to predict elastic response of cellular materials.

The theories developed by Twiss and Eringen [1], [2] are enough to have a complete Eulerian description of the mixtures with microstructure. In the framework of these theories some uniqueness, continuous dependence and stability results have been established by Ghiba in the papers [8], [9].

According to Ieşan [10], [11], the Eulerian description is appropriate to model the behavior of fluid mixtures, while for mixtures of solids it is natural to use the Lagrangian description. In fact, these two descriptions lead to different theories. Moreover, when the Lagrangian description is used to develop a theory of mixtures for solids, then the initial boundary value problems associated are formulated in a natural way and the boundary conditions have a clear physical and mathematical meaning.

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In recent years a great attention has been given to the theory of mixtures developed in Lagrangian description. A special attention has been paid to include some terms in the basic formulation of the theory in order to reflect the microstructure of the constituents (see for example the works of Ieşan [11], [12], [13], Galeş [14], [16] and Chiriţă and Galeş [15]).

In the paper by Galeş [14], a nonlinear theory for binary mixtures of micropolar thermoelastic solids has been introduced. Following Elangovan et al. [7], mixtures of micropolar thermoelastic solids are useful to predict elastic response of cellular materials.

In the present work we consider the linearized theory presented by Galeş [14] and we treat the uniqueness and the continuous data dependence problems with no definiteness assumptions on the internal energy. Thus, the results hold for the entire class of micropolar thermoelastic mixtures. The method adopted here is based on Lagrange identity and logarithmic convexity arguments (see [17], [18], [19] and the references therein for applications of the method in classical mathematical models).

The results of the present paper prove that in the motion following any sufficiently small changes in the external data system, the solution of the initial–boundary value problem is everywhere arbitrary small in magnitude. This means that the approach of the linear theory derived in [14] is well posed against errors in the external given data.

II. FORMULATION OF THE PROBLEM

We consider a body that in the reference configuration taken at time $t = 0$, occupies the bounded regular region B of the Euclidian three-dimensional space and assume that its boundary ∂B is a piecewise smooth surface. A binary mixture of micropolar thermoelastic solids fills B.

We refer the motion of the body to a fixed system of rectangular Cartesian axes. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers $(1, 2, 3)$, summation over repeated subscripts is implied, subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, and a superposed dot denotes time differentiation. Greek indices understood to range over the integers $(1, 2)$ and summation convention are not used for these indices.

According to the linear theory, the behavior of the mixture is characterized by the displacement vectors $\mathbf{u}$ and $\mathbf{w}$, the microrotation vectors $\boldsymbol{\varphi}^{(1)}$ and $\boldsymbol{\varphi}^{(2)}$ associated with the two constituents and the temperature T. As discussed in [14], the

fundamental system of field equations that governs the motion of an anisotropic mixture consists of:
–the equations of motion

$$
\begin{aligned}
&t_{ji,j} - p_i + \rho_1^0 F_i^{(1)} = \rho_1^0 \ddot{u}_i, \\
&s_{ji,j} + p_i + \rho_2^0 F_i^{(2)} = \rho_2^0 \ddot{w}_i, \\
&m_{ji,j}^{(1)} + \epsilon_{ijk}[t_{jk} + s_{jk}] - \mathcal{R}_i + \rho_1^0 G_i^{(1)} = \rho_1^0 I_{ij}^{(1)} \ddot{\varphi}_j^{(1)}, \\
&m_{ji,j}^{(2)} + \mathcal{R}_i + \rho_2^0 G_i^{(2)} = \rho_2^0 I_{ij}^{(2)} \ddot{\varphi}_j^{(2)},
\end{aligned} \tag{1}
$$

in $B \times (0,\infty)$, where ρ_1^0 and ρ_2^0 are the densities at time $t=0$, $I_{ij}^{(1)}$ and $I_{ij}^{(2)}$ are the coefficients of inertia at time $t=0$, t_{ij} and s_{ij} are the partial stress tensors, $m_{ij}^{(1)}$ and $m_{ij}^{(2)}$ are the partial couple stress tensors, p_i is the internal body force, $\mathcal{R}_i$ is the internal body couple, $F_i^{(1)}$ and $F_i^{(2)}$ are the body forces and $G_i^{(1)}$ and $G_i^{(2)}$ are the body couples;
–the energy equation

$$
\rho_0 T_0 \dot{\eta} = q_{i,i} + \rho_0 r, \tag{2}
$$

in $B \times (0,\infty)$, where $\rho_0 = \rho_1^0 + \rho_2^0$, T_0 is the constant absolute temperature of the mixture in the reference configuration, q_i is the heat flux vector, η is the entropy and r is the heat supply;
–the constitutive equations

$$
\begin{aligned}
t_{ij} =\ & A_{ijrs}e_{rs} + B_{ijrs}g_{rs} + F_{ijrs}^{(1)}\gamma_{rs}^{(1)} + F_{ijrs}^{(2)}\gamma_{rs}^{(2)} \\
& + a_{ijk}d_k + b_{ijk}\pi_k - \alpha_{ij}^* T, \\
s_{ij} =\ & B_{rsij}e_{rs} + C_{ijrs}g_{rs} + H_{ijrs}^{(1)}\gamma_{rs}^{(1)} + H_{ijrs}^{(2)}\gamma_{rs}^{(2)} \\
& + c_{ijk}d_k + d_{ijk}\pi_k - \beta_{ij}^* T, \\
m_{ij}^{(1)} =\ & F_{rsij}^{(1)}e_{rs} + H_{rsij}^{(1)}g_{rs} + D_{ijrs}^{(1)}\gamma_{rs}^{(1)} + D_{ijrs}^{(3)}\gamma_{rs}^{(2)} \\
& + \alpha_{ijk}^{(1)}d_k + \beta_{ijk}^{(1)}\pi_k - \nu_{ij}^* T, \\
m_{ij}^{(2)} =\ & F_{rsij}^{(2)}e_{rs} + H_{rsij}^{(2)}g_{rs} + D_{rsij}^{(3)}\gamma_{rs}^{(1)} + D_{ijrs}^{(2)}\gamma_{rs}^{(2)} \\
& + \alpha_{ijk}^{(2)}d_k + \beta_{ijk}^{(2)}\pi_k - \mu_{ij}^* T, \\
p_i =\ & a_{jki}e_{jk} + c_{jki}g_{jk} + \alpha_{jki}^{(1)}\gamma_{jk}^{(1)} + \alpha_{jki}^{(2)}\gamma_{jk}^{(2)} \\
& + a_{ij}d_j + c_{ij}\pi_j - \tau_i^* T, \\
\mathcal{R}_i =\ & b_{jki}e_{jk} + d_{jki}g_{jk} + \beta_{jki}^{(1)}\gamma_{jk}^{(1)} + \beta_{jki}^{(2)}\gamma_{jk}^{(2)} \\
& + c_{ji}d_j + b_{ij}\pi_j - \sigma_i^* T, \\
\rho_0 \eta =\ & \alpha_{ij}^* e_{ij} + \beta_{ij}^* g_{ij} + \nu_{ij}^* \gamma_{ij}^{(1)} + \mu_{ij}^* \gamma_{ij}^{(2)} \\
& + \tau_i^* d_i + \sigma_i^* \pi_i + a^* T, \\
q_i =\ & k_{ij}T_{,j},
\end{aligned} \tag{3}
$$

in $\overline{B} \times [0,\infty)$, where e_{ij}, g_{ij}, $\gamma_i^{(\alpha)}$, π_i are defined by
–the geometric equations

$$
\begin{aligned}
&e_{ji} = u_{i,j} + \epsilon_{sij}\varphi_s^{(1)}, \qquad g_{ji} = w_{i,j} + \epsilon_{sij}\varphi_s^{(1)}, \\
&d_i = u_i - w_i, \qquad \gamma_{ij}^{(\alpha)} = \varphi_{j,i}^{(\alpha)}, \qquad \pi_i = \varphi_i^{(1)} - \varphi_i^{(2)}.
\end{aligned} \tag{4}
$$

The constitutive coefficients have the following symmetries

$$
\begin{aligned}
A_{ijrs} = A_{rsij}, \quad C_{ijrs} = C_{rsij}, \\
D_{ijrs}^{(\alpha)} = D_{rsij}^{(\alpha)}, \quad a_{ij} = a_{ji}, \quad b_{ij} = b_{ji},
\end{aligned} \tag{5}
$$

and moreover, the dissipation inequality implies

$$
\Phi \equiv \frac{1}{T_0} k_{ij} T_{,i} T_{,j} \geq 0. \tag{6}
$$

To the above equations we have to adjoin boundary conditions and initial conditions. We consider the following boundary conditions:

$$
\begin{aligned}
&u_i = \widetilde{u}_i, \quad w_i = \widetilde{w}_i, \quad \varphi_i^{(1)} = \widetilde{\varphi}_i^{(1)}, \\
&\varphi_i^{(2)} = \widetilde{\varphi}_i^{(2)}, \quad T = \widetilde{T} \quad \text{on } \partial B \times [0,\infty)
\end{aligned} \tag{7}
$$

where $\widetilde{u}_i$, $\widetilde{w}_i$, $\widetilde{\varphi}_i^{(\alpha)}$ and $\widetilde{T}$ are given.

We denote by $(\mathcal{P})$ the initial boundary value problem defined by Eqs (1)–(4) and (7) and the initial conditions

$$
\begin{aligned}
&u_i(\mathbf{x},0) = a_i^0(\mathbf{x}), \quad w_i(\mathbf{x},0) = b_i^0(\mathbf{x}), \\
&\dot{u}_i(\mathbf{x},0) = c_i^0(\mathbf{x}), \quad \dot{w}_i(\mathbf{x},0) = f_i^0(\mathbf{x}), \\
&\varphi_i^{(\alpha)}(\mathbf{x},0) = \psi_i^{(\alpha)}(\mathbf{x}), \quad \dot{\varphi}_i^{(\alpha)}(\mathbf{x},0) = \omega_i^{(\alpha)}(\mathbf{x}), \\
&T(\mathbf{x},0) = \theta(\mathbf{x}), \qquad \mathbf{x} \in \overline{B},
\end{aligned} \tag{8}
$$

where $a_i^0, b_i^0, c_i^0, f_i^0, \psi_i^{(\alpha)}, \omega_i^{(\alpha)}$ and θ are prescribed functions.

III. PRELIMINARY RESULTS

In this section we derive some Lagrange identities useful in the study of uniqueness and continuous dependence problems. Let us denote by W the internal energy density, that is,

$$
\begin{aligned}
W =\ & \frac{1}{2}A_{ijrs}e_{ij}e_{rs} + B_{ijrs}e_{ij}g_{rs} + \frac{1}{2}C_{ijrs}g_{ij}g_{rs} \\
& + \sum_{\alpha=1}^{2}\left(\frac{1}{2}D_{ijrs}^{(\alpha)}\gamma_{ij}^{(\alpha)}\gamma_{rs}^{(\alpha)} + F_{ijrs}^{(\alpha)}e_{ij}\gamma_{rs}^{(\alpha)} + H_{ijrs}^{(\alpha)}g_{ij}\gamma_{rs}^{(\alpha)}\right) \\
& + D_{ijrs}^{(3)}\gamma_{ij}^{(1)}\gamma_{rs}^{(2)} + \frac{1}{2}a_{ij}d_i d_j + \frac{1}{2}b_{ij}\pi_i\pi_j \\
& + c_{ij}d_i\pi_j + a_{ijk}e_{ij}d_k + b_{ijk}e_{ij}\pi_k \\
& + c_{ijk}g_{ij}d_k + d_{ijk}g_{ij}\pi_k + \alpha_{ijk}^{(1)}\gamma_{ij}^{(1)}d_k \\
& + \beta_{ijk}^{(1)}\gamma_{ij}^{(1)}\pi_k + \alpha_{ijk}^{(2)}\gamma_{ij}^{(2)}d_k + \beta_{ijk}^{(2)}\gamma_{ij}^{(2)}\pi_k,
\end{aligned} \tag{9}
$$

and let us introduce the kinetic energy $\mathcal{K}$, the internal energy $\mathcal{W}$, the dissipation energy Δ, the thermal energy $\mathcal{T}$, the total energy $\mathcal{E}$, namely

$$
\begin{aligned}
\mathcal{K}(t) = \frac{1}{2}\int_B \Big(&\rho_1^0 \dot{u}_i(t)\dot{u}_i(t) + \rho_2^0 \dot{w}_i(t)\dot{w}_i(t) \\
& + \rho_1^0 I_{ij}^{(1)}\dot{\varphi}_i^{(1)}(t) + \rho_2^0 I_{ij}^{(2)}\dot{\varphi}_i^{(2)}(t)\Big)dv,
\end{aligned} \tag{10}
$$

$$
\mathcal{W}(t) = \int_B W(t)dv, \tag{11}
$$

$$
\Delta(t) = \int_0^t \int_B \Phi(\tau)dv\,d\tau, \tag{12}
$$

$$
\mathcal{T}(t) = \frac{1}{2}\int_B a^* T^2(t)dv, \tag{13}
$$

$$
\mathcal{E}(t) = \mathcal{K}(t) + \mathcal{W}(t) + \Delta(t) + \mathcal{T}(t), \tag{14}
$$

and the functions

$$
\begin{aligned}
I(t) = \frac{1}{2}\int_B \Big[&\rho_1^0 u_i(t)u_i(t) + \rho_2^0 w_i(t)w_i(t) \\
& + \rho_1^0 I_{ij}^{(1)}\varphi_i^{(1)}(t)\varphi_j^{(1)}(t) + \rho_2^0 I_{ij}^{(2)}\varphi_i^{(2)}(t)\varphi_j^{(2)}(t)\Big]dv \\
& + \frac{1}{2}\int_0^t \int_B \Big[\frac{1}{T_0}k_{ij}\Big(\int_0^\tau T_{,i}(s)ds\Big)\Big(\int_0^\tau T_{,j}(s)ds\Big)\Big]dvd\tau,
\end{aligned} \tag{15}
$$

$$P(t,\tau) = \int_B \Big[\rho_1^0 F_i^{(1)}(t)\dot{u}_i(\tau) + \rho_2^0 F_i^{(2)}(t)\dot{w}_i(\tau)$$

$$+ \sum_{\alpha=1,2} \rho_\alpha^0 G_i^{(\alpha)}(t)\dot{\varphi}_i^{(\alpha)}(\tau) + \frac{\rho_0 r(\tau)}{T_0} T(t) \Big] dv$$

$$+ \int_{\partial B} \Big[t_{ji}(t)\dot{u}_i(\tau) + s_{ji}(t)\dot{w}_i(\tau)$$

$$+ \sum_{\alpha=1,2} m_{ji}^{(\alpha)}(t)\dot{\varphi}_i^{(\alpha)}(\tau) + \frac{q_j(\tau)}{T_0} T(t) \Big] n_j da, \tag{16}$$

$$Q(t,\tau) = \int_B \Big[\rho_1^0 F_i^{(1)}(t) u_i(\tau) + \rho_2^0 F_i^{(2)}(t) w_i(\tau)$$

$$+ \sum_{\alpha=1,2} \rho_\alpha^0 G_i^{(\alpha)}(t)\varphi_i^{(\alpha)}(\tau) + \frac{\rho_0 T(t)}{T_0} \int_0^\tau r(s)ds \Big] dv$$

$$+ \int_{\partial B} \Big[t_{ji}(t) u_i(\tau) + s_{ji}(t) w_i(\tau)$$

$$+ \sum_{\alpha=1,2} m_{ji}^{(\alpha)}(t)\varphi_i^{(\alpha)}(\tau) + \frac{T(t)}{T_0} \int_0^\tau q_j(s)ds \Big] n_j da, \tag{17}$$

where n_i are the components of the outward unit normal vector.

Lemma 1 (Conservation law of total energy) *If* $\mathcal{U} = \{\mathbf{u}, \mathbf{w}, \boldsymbol{\varphi}^{(1)}, \boldsymbol{\varphi}^{(2)}, T\}$ *is solution of the problem* $(\mathcal{P})$, *then the following conservation energy law holds:*

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t P(\tau,\tau)d\tau. \tag{18}$$

Proof. In view of the (3), (5) and (9) we deduce

$$t_{ij}\dot{e}_{ij} + s_{ij}\dot{g}_{ij} + m_{ij}^{(1)}\dot{\gamma}_{ij}^{(1)} + m_{ij}^{(2)}\dot{\gamma}_{ij}^{(2)} + p_i\dot{d}_i$$
$$+ \mathcal{R}_i \pi_i + \rho_0 \dot{\eta} T = \frac{\partial}{\partial t}\Big(\frac{1}{2}a^* T^2 + W \Big). \tag{19}$$

On the other hand, in view of (1), (2) and (4) we obtain

$$t_{ij}\dot{e}_{ij} + s_{ij}\dot{g}_{ij} + m_{ij}^{(1)}\dot{\gamma}_{ij}^{(1)} + m_{ij}^{(2)}\dot{\gamma}_{ij}^{(2)} + p_i\dot{d}_i + \mathcal{R}_i\pi_i + \rho_0\dot{\eta}T$$

$$= \Big(t_{ji}\dot{u}_i + s_{ji}\dot{w}_i + m_{ji}^{(1)}\dot{\varphi}_i^{(1)} + m_{ji}^{(2)}\dot{\varphi}_i^{(2)} + \frac{1}{T_0}q_j T \Big)_{,j}$$

$$+ \rho_1^0 F_i^{(1)}\dot{u}_i + \rho_2^0 F_i^{(2)}\dot{w}_i + \rho_1^0 G_i^{(1)}\dot{\varphi}_i^{(1)} + \rho_2^0 G_i^{(2)}\dot{\varphi}_i^{(2)}$$

$$+ \frac{1}{T_0}\rho_0 rT - \frac{1}{2}\frac{\partial}{\partial t}\Big(\rho_1^0 \dot{u}_i\dot{u}_i + \rho_2^0 \dot{w}_i\dot{w}_i$$

$$+ \rho_1^0 I_{ij}^{(1)}\dot{\varphi}_i^{(1)}\dot{\varphi}_j^{(1)} + \rho_2^0 I_{ij}^{(2)}\dot{\varphi}_i^{(2)}\dot{\varphi}_j^{(2)} \Big) - \frac{1}{T_0}k_{ij}T_{,i}T_{,j}. \tag{20}$$

Then (19) and (20) imply

$$\frac{1}{2}\frac{\partial}{\partial t}\Big(\rho_1^0 \dot{u}_i\dot{u}_i + \rho_2^0 \dot{w}_i\dot{w}_i + \rho_1^0 I_{ij}^{(1)}\dot{\varphi}_i^{(1)}\dot{\varphi}_j^{(1)}$$

$$+ \rho_2^0 I_{ij}^{(2)}\dot{\varphi}_i^{(2)}\dot{\varphi}_j^{(2)} + a^* T^2 + 2W \Big) + \frac{1}{T_0}k_{ij}T_{,i}T_{,j} =$$

$$\rho_1^0 F_i^{(1)}\dot{u}_i + \rho_2^0 F_i^{(2)}\dot{w}_i + \rho_1^0 G_i^{(1)}\dot{\varphi}_i^{(1)} + \rho_2^0 G_i^{(2)}\dot{\varphi}_i^{(2)} + \frac{1}{T_0}\rho_0 rT$$

$$+ \frac{1}{2}\Big[(t_{ji} + s_{ji})(\dot{u}_i + \dot{w}_i) + (t_{ji} - s_{ji})(\dot{u}_i - \dot{w}_i)$$

$$+ (m_{ji}^{(1)} + m_{ji}^{(2)})(\dot{\varphi}_i^{(1)} + \dot{\varphi}_i^{(1)})$$

$$+ (m_{ji}^{(1)} - m_{ji}^{(2)})(\dot{\varphi}_i^{(1)} - \dot{\varphi}_i^{(2)}) + \frac{2}{T_0}q_j T \Big]_{,j}. \tag{21}$$

By an integration of above relation over $B \times [0,t]$ and by using the divergence theorem and the relations (11)–(14) and (16) we obtain the identity (18) and the proof is complete.

Lemma 2 *Let* $\mathcal{U} = \{\mathbf{u}, \mathbf{w}, \boldsymbol{\varphi}^{(1)}, \boldsymbol{\varphi}^{(2)}, T\}$ *be solution of the problem* $(\mathcal{P})$. *Then for every* $t \in [0,\infty)$, *the following identity holds*

$$\frac{dI}{dt}(t) = \frac{dI}{dt}(0) + 2\int_0^t [\mathcal{K}(\tau) - \mathcal{W}(\tau) - \mathcal{T}(\tau)]d\tau$$

$$+ \int_0^t Q(\tau,\tau)d\tau + \int_0^t \int_B \rho_0 \eta(0)T(\tau)dvd\tau. \tag{22}$$

Proof. It follows from (3) and (9) that

$$t_{ij}e_{ij} + s_{ij}g_{ij} + p_i d_i + m_{ij}^{(1)}\gamma_{ij}^{(1)} + m_{ij}^{(2)}\gamma_{ij}^{(2)}$$
$$+ \mathcal{R}_i\pi_i + \rho_0\eta T = 2W + a^* T^2. \tag{23}$$

By taking into account the relations (1), (2) and (4) we obtain

$$t_{ij}e_{ij} + s_{ij}g_{ij} + p_i d_i + m_{ij}^{(1)}\gamma_{ij}^{(1)} + m_{ij}^{(2)}\gamma_{ij}^{(2)} + \mathcal{R}_i\pi_i + \rho_0\eta T =$$

$$= -\frac{\partial}{\partial t}[\rho_1^0 u_i\dot{u}_i + \rho_2^0 w_i\dot{w}_i + \rho_1^0 I_{ij}^{(1)}\varphi_i^{(1)}\dot{\varphi}_i^{(1)} + \rho_2^0 I_{ij}^{(2)}\varphi_i^{(2)}\dot{\varphi}_i^{(2)}$$

$$+ \frac{1}{2T_0}k_{ij}\int_0^t T_{,i}d\tau \int_0^t T_{,j}d\tau] + \rho_1^0\dot{u}_i\dot{u}_i + \rho_2^0\dot{w}_i\dot{w}_i$$

$$+ \rho_1^0 I_{ij}^{(1)}\dot{\varphi}_i^{(1)}\dot{\varphi}_i^{(1)} + \rho_2^0 I_{ij}^{(2)}\dot{\varphi}_i^{(2)}\dot{\varphi}_i^{(2)}$$

$$+ \rho_1^0 F_i^{(1)} u_i + \rho_2^0 F_i^{(2)} w_i + \rho_1^0 G_i^{(1)}\varphi_i^{(1)} + \rho_2^0 G_i^{(2)}\varphi_i^{(2)}$$

$$+ \frac{T}{T_0}\int_0^t \rho_0 r d\tau + \rho_0\eta(0)T$$

$$+ [t_{ij}u_j + s_{ij}w_j + m_{ij}^{(1)}\varphi_j^{(1)} + m_{ij}^{(2)}\varphi_j^{(2)} + \frac{T}{T_0}\int_0^t q_i d\tau]_{,i}. \tag{24}$$

In view of the above relations, we obtain an identity which integrated over $B \times [0,t]$ leads with the help of the divergence theorem and relations (10)–(15) and (17) to (22) and the proof is complete.

The above lemmas yield the following:

Remark 1 *Assume that* $\mathcal{U} = \{\mathbf{u}, \mathbf{w}, \boldsymbol{\varphi}^{(1)}, \boldsymbol{\varphi}^{(2)}, T\}$ *is solution of the problem* $(\mathcal{P})$. *Then for every* $t \in [0,\infty)$, *we have*

$$\frac{dI}{dt}(t) = \frac{dI}{dt}(0) + \int_0^t [4\mathcal{K}(\tau) + 2\Delta(\tau)]d\tau - 2t\mathcal{E}(0)$$

$$- 2\int_0^t \int_0^\tau P(s,s)dsd\tau + \int_0^t Q(\tau,\tau)d\tau \tag{25}$$

$$+ \int_0^t \int_B \rho_0\eta(0)T(\tau)dvd\tau.$$

IV. Uniqueness result

Theorem 1 *Suppose that* ρ_1^0, ρ_2^0 *and* a^* *are strictly positive and* $I_{ij}^{(1)}$, $I_{ij}^{(2)}$ *and* k_{ij} *are symmetric and positive definite. Then, there is at most one solution of the problem* $(\mathcal{P})$.

Proof. Due to the linearity of the problem it suffices to prove that the only solution corresponding to vanishing external given data also vanishing. Thus, let $\mathcal{P}_0$ be the initial boundary value problem with vanishing external data. Then, from relations (14)–(17), we get

$$I(0) = 0, \quad \dot{I}(0) = 0, \quad \mathcal{E}(0) = 0,$$
$$P(t,\tau) = 0, \quad Q(t,\tau) = 0, \quad t \geq 0, \quad \tau \geq 0. \tag{26}$$

Moreover, it suffices to prove the uniqueness on an interval $[0, \bar{t}]$, $\bar{t} > 0$, in order to assure the uniqueness on $[0, \infty)$.

Since $I(0) = 0$ and I is continuous on $[0, \infty)$, then there exist $\bar{t} > 0$ such that

$$\int_0^{\bar{t}} I(t)dt < \infty. \tag{27}$$

Introducing the function

$$G(t) = \int_0^t I(\tau)d\tau, \qquad t \in [0, \bar{t}], \tag{28}$$

then, from (15) and (26) we deduce

$$\dot{G}(t) = \int_0^t \dot{I}(\tau)d\tau = \int_0^t \int_B \Big[\rho_1^0 \dot{u}_i u_i + \rho_2^0 \dot{w}_i w_i$$
$$+ \rho_1^0 I_{ij}^{(1)} \dot{\varphi}_j^{(1)} \varphi_i^{(1)} + \rho_2^0 I_{ij}^{(2)} \dot{\varphi}_j^{(2)} \varphi_i^{(2)} \Big] dvd\tau \tag{29}$$
$$+ \int_0^t \int_0^\tau \int_B \Big[\frac{1}{T_0} k_{ij} T_{,i}(\tau) \Big(\int_0^s T_{,j}(z)dz \Big) \Big] dvdsd\tau.$$

Further, from (25), (26) and (28) we have

$$\ddot{G}(t) = \dot{I}(t) = \int_0^t \Big[4\mathcal{K}(\tau) + 2\Delta(\tau) \Big] d\tau. \tag{30}$$

Substituting I in (28) and $\mathcal{K}$, Δ in (30), then from (28), (29), (30) and the Schwarz inequality we deduce

$$G(t)\ddot{G}(t) - \dot{G}^2(t) \geq 0, \qquad t \in [0, \bar{t}]. \tag{31}$$

The relation (31) assures that $G(t) = 0$ for every $t \in [0, \bar{t}]$. Indeed, if there exist $t^* \in (0, \bar{t})$ such that $G(t^*) > 0$, then there exists an open interval (t_1, t_2) on which $G(t) > 0$. So for $0 \leq t_1 < t < t_2 \leq \bar{t}$ we may divide (31) by G^2 to obtain

$$\frac{d^2}{dt^2}[\ln G(t)] \geq 0. \tag{32}$$

Jensen's inequality together with the continuity of G then gives

$$G(t) \leq [G(t_1)]^{(t_2-t)/(t_2-t_1)}[G(t_2)]^{(t-t_1)/(t_2-t_1)}. \tag{33}$$

Now, since $G(t_1) = 0$, it follows from (33) that $G(t)$ vanishes identically on $[t_1, t_2]$ and hence for $0 \leq t \leq \bar{t}$. This fact clearly implies the uniqueness of the solution $\mathcal{U}$ to the problem $(\mathcal{P})$.

V. Continuous dependence

In this section we shall denote by $(\mathcal{P}_f)$ the boundary value problem corresponding to homogeneous boundary data, homogeneous initial data and $r = 0$.

Theorem 2 (Continuous dependence upon body loads). *Assume that ρ_1^0, ρ_2^0 and a^* are strictly positive and $I_{ij}^{(1)}$, $I_{ij}^{(2)}$ and k_{ij} are symmetric and positive definite. If $\bar{t} \in [0, \infty)$ is such that (27) holds true, and $G^*(t)$, $t \in [0, \bar{t}]$, is defined by*

$$G^*(t) = \int_0^t I(\tau)d\tau + \frac{1}{2}\bar{t}^4 \int_0^{\bar{t}} \|\mathcal{F}(\tau)\|_m^2 d\tau, \tag{34}$$

where

$$\|\mathcal{F}\|_m^2 = \int_B \Big[\rho_1^0 F_i^{(1)} F_i^{(1)} + \rho_2^0 F_i^{(2)} F_i^{(2)}$$
$$+ \frac{\rho_1^0}{I_1^m} G_i^{(1)} G_i^{(1)} + \frac{\rho_2^0}{I_2^m} G_i^{(2)} G_i^{(2)} \Big] dv, \tag{35}$$

I_α^m is the lowest eigenvalue of the tensor $I_{ij}^{(\alpha)}$. Then, for the solution $\mathcal{U}$ of the problem $(\mathcal{P}_f)$ we have the following estimate

$$G^*(t) \leq e^{\delta(1-\delta)}[G^*(\bar{t})]^\delta[G^*(0)]^{1-\delta}, \ \delta = t/\bar{t}. \tag{36}$$

Proof. Let us introduce the notation

$$\|\mathcal{U}\|^2 = \int_B \Big[\rho_1^0 u_i u_i + \rho_2^0 w_i w_i$$
$$+ \rho_1^0 I_{ij}^{(1)} \varphi_i^{(1)} \varphi_j^{(1)} + \rho_2^0 I_{ij}^{(2)} \varphi_i^{(2)} \varphi_j^{(2)} \Big] dv. \tag{37}$$

Using similar arguments as those utilized in the proof of the Theorem 1, we deduce

$$G^*\ddot{G}^* - [\dot{G}^*]^2 \geq G^* \Big[\int_0^t Q(\tau, \tau)d\tau$$
$$- 2 \int_0^t (t-\tau)P(\tau, \tau)d\tau \Big] + \bar{t}^4 \Big(\int_0^{\bar{t}} \|\mathcal{F}(\tau)\|_m^2 d\tau \Big) \tag{38}$$
$$\times \Big(\int_0^t \|\dot{\mathcal{U}}(\tau)\|^2 d\tau + \int_0^t \int_0^\tau \int_B \frac{1}{T_0} k_{ij} T_{,j} T_{,i} dvdsd\tau \Big).$$

Applying Schwarz's inequality and the arithmetic–geometric mean inequality, after some calculations, we get

$$-G^* \int_0^t Q(\tau, \tau)d\tau \leq [G^*]^2 \bar{t}^{-2}, \tag{39}$$

$$2G^* \int_0^t (t-\tau)P(\tau, \tau)d\tau \leq [G^*]^2 \bar{t}^{-2}$$
$$+ \bar{t}^4 \Big(\int_0^t \|\mathcal{F}(\tau)\|_m^2 d\tau \Big) \Big(\int_0^t \|\dot{\mathcal{U}}(\tau)\|^2 d\tau \Big). \tag{40}$$

The relations (38), (39) and (40) imply that

$$G^*\ddot{G}^* - [\dot{G}^*]^2 \geq -2[G^*]^2 \bar{t}^{-2}. \tag{41}$$

Dividing (41) by $[G^*]^2$ we obtain

$$\frac{d^2}{dt^2}[\ln G^*(t)] \geq -2\bar{t}^{-2}, \tag{42}$$

which yields (36).

VI. Concluding remarks

In this paper we utilized the identity (25) and logarithmic convexity arguments in order to prove the uniqueness of solution and to examine the question of continuous dependence on the body forces and body couples. The results are proved with no definiteness assumptions on the internal energy. Following similar techniques as those presented in [17], [18], [19] it is easy to examine the question of continuous dependence on the heat supply or on the initial data or on the boundary data.

Clearly, when the internal energy density W is a positive semi–definite quadratic form, then the conservation law of energy (18) and the Gronwall's lemma lead to a Liapounov stability theorem.

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