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On the Approximate Solution of a Nonlinear Singular Integral Equation

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Abstract**—**In this study, the existence and uniqueness of the solution of a nonlinear singular integral equation that is defined on a region in the complex plane is proven and a method is given for finding the solution.

Keywords**—**Approximate solution, Fixed-point principle, Nonlinear singular integral equations, Vekua integral operator

I. INTRODUCTION

As it is known, the application area of nonlinear singular integral equation is so extensive such as the theories of elasticity, viscoelasticity, thermo elasticity, hydrodynamics, fluid mechanics and mathematical physics and many other fields [1]-[6].

Furthermore, the solution of the seismic wave equation that has a great importance in elastodynamics is investigated by reducing it to the solution of nonlinear singular integral equation by using Hilbert transformation [7].

All of these studies contribute to update of investigations about the solution of the nonlinear singular integral equation by using approximate and constructive methods.

In this study, the approximate solution of nonlinear singular integral equations that is defined on a bounded region in complex plane is discussed.

It is known that the investigation of the problems of the Dirichlet boundary-value problems for many nonlinear differential equation systems which have partial derivatives and defined on a planar region $G \subset \mathbb{C}$ can be reduced to the investigation of the problem of a nonlinear singular integral equation which is in the following form [8], [9], [10]-[14]

$$\varphi(z) = \lambda.F(z,\varphi(z),T_G f(.,\varphi(.))(z),\Pi_G g(.,\varphi(.))(z)). \tag{1}$$

Here, let

$$D_0 = \left\{(z,\varphi) : z \in \overline{G} = \partial G \cup \overset{\circ}{G}, \varphi \in \mathbb{C}\right\} = \overline{G} \times \mathbb{C}$$

∂G is the boundary of the region G, $\overset{\circ}{G}$ is the set of interior points of G and $D = \{(z,\varphi,t,s) : z \in \overline{G}, \varphi,t,s \in \mathbb{C}\} = \overline{G} \times \mathbb{C}^3$ are as given, $f, g : D_0 \to \mathbb{C}$ and $F : D \to \mathbb{C}$ are the known functions, $\lambda \in \mathbb{R}$ is any scalar parameter, $h \in H_\alpha(\overline{G})(0 < \alpha < 1)$ and for $z = x+iy, \zeta = \xi + i\eta$ followings are the known Vekua integral operators [15].

$$T_G h(.)(z) = (-1/\pi).\iint_G h(\zeta).(\zeta - z)^{-1} d\xi d\eta,$$

$$\Pi_G h(.)(z) = (-1/\pi)\iint_G h(\zeta).(\zeta - z)^{-2} d\xi d\eta$$

In this study, by taking base, a more useful modified variant of Schauder and Banach fixed point principles, the existence and uniqueness of the solution of the Equation (1) is proven with more weak conditions on the functions f, g and F.

Furthermore, iteration is given for the approximate solution of the Equation (1) and it is shown that the iteration is converging to the real solution.

II. BASIC ASSUMPTIONS AND AUXILIARY RESULTS

Further, throughout the study, if not the opposite said we take the set $G \subset \mathbb{C}$ as bounded and simple connected region. For $\overset{\circ}{G}$ is to be the set of interior points of the region G and ∂G is to be the boundary of G let $\overline{G} = \partial G \cup \overset{\circ}{G}$.

If, for every $z_1, z_2 \in \overline{G}$ there exist $H > 0$ and $\alpha \in (0,1]$ numbers such that

$$|\varphi(z_1) - \varphi(z_2)| \le H.|z_1 - z_2|^\alpha$$

then it is said that the function $\varphi : \overline{G} \to \mathbb{C}$ satisfies the Holder condition on the set $\overline{G}$ with exponent α. We will show the set of all functions that satisfies Holder condition on the set $\overline{G}$ with exponent α with $H_\alpha(\overline{G})$.

For $\varphi \in H_\alpha(\overline{G})$ and $0 < \alpha < 1$, the vector space $\left(H_\alpha(\overline{G}); \|.\|_\alpha\right)$ is a Banach space with the norm

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$$\|\varphi\|_{\alpha} = \|\varphi\|_{H_\alpha(\overline{G})} \equiv \|\varphi\|_{\infty} + H(\varphi,\alpha;\overline{G}).$$

Here,

$$\|\varphi\|_{\infty} = \max\{|\varphi(z)| : z \in \overline{G}\},$$
$$H(\varphi,\alpha;\overline{G}) = \sup_{z_1,z_2\in\overline{G}, z_1\neq z_2} \{|\varphi(z_1) - \varphi(z_2)|\}$$

are as given.

For $L_p(\overline{G}) = \left\{\varphi : \overline{G} \to \mathbb{C} : \iint_G |\varphi(\zeta)|^p \, d\xi d\eta < +\infty\right\}$, $1 < p$, the vector space $\left(L_p(\overline{G}); \|.\|_p\right)$ is a Banach space with the following norm,

$$\|\varphi\|_p = \|\varphi\|_{L_p(\overline{G})} \equiv \left(\iint_G |\varphi(\zeta)|^p \, d\xi d\eta\right)^{1/p}.$$

Further, throughout the study we take $D_0 = \{(z,\varphi) : z \in \overline{G}, \varphi \in \mathbb{C}\} = \overline{G} \times \mathbb{C}$ and $D = \{(z,\varphi,t,s) : z \in \overline{G}, \varphi,t,s \in \mathbb{C}\} = \overline{G} \times \mathbb{C}^3$. Besides, for every $z_1, z_2 \in \overline{G}$ and $(z_k,\varphi_k) \in D_0$, $(z_k,\varphi_k,t_k,s_k) \in D$, for $k = 1,2$ we suppose that the number $\alpha \in (0,1)$ and the positive constants $m_1, m_2, n_1, n_2, l_1, l_2, l_3, l_4$ are exist such that following inequalities be satisfied

$$|f(z_1,\varphi_1) - f(z_2,\varphi_2)| \leq m_1|z_1 - z_2|^{\alpha} + m_2|\varphi_1 - \varphi_2| \qquad (2)$$

$$|g(z_1,\varphi_1) - g(z_2,\varphi_2)| \leq n_1|z_1 - z_2|^{\alpha} + n_2|\varphi_1 - \varphi_2| \qquad (3)$$

$$|F(z_1,\varphi_1,t_1,s_1) - F(z_2,\varphi_2,t_2,s_2)| \leq l_1|z_1 - z_2|^{\alpha} + l_2|\varphi_1 - \varphi_2| + l_3|t_1 - t_2| + l_4|s_1 - s_2| \qquad (4)$$

We will denote the set of functions that satisfy the conditions (2), (3) and (4) with $H_{\alpha,1}(m_1,m_2;D_0), H_{\alpha,1}(n_1,n_2;D_0)$ and $H_{\alpha,1,1,1}(l_1,l_2,l_3,l_4;D)$ respectively.

Now, let us give some supplementary lemmas for the theorems about the existence and uniqueness of the solution of the Equation (1).

Lemma 2.1. If $\varphi \in H_\alpha(\overline{G})$, $\alpha \in (0,1)$ *then for every* $p > 1$ *and* $\varepsilon \in (0,d)$ *the following inequality is held*

$$\|\varphi\|_{\infty} \leq 2.\varepsilon^{\alpha}.\|\varphi\|_{\alpha} + (\pi\varepsilon^2)^{-1/p}.\|\varphi\|_p, \qquad (5)$$

where $d = \sup\{|z_1 - z_2| : z_1, z_2 \in G\}$.

Lemma 2.2. For $\varphi \in H_\alpha(\overline{G})$, $\alpha \in (0,1)$ *and* $p > 1$ *the following inequality is held*

$$\|\varphi\|_{\infty} \leq M(\alpha,p).\|\varphi\|_{\alpha}^{\frac{2}{2+\alpha p}}.\|\varphi\|_{p}^{\frac{\alpha p}{2+\alpha p}}. \qquad (6)$$

Here, $M(\alpha,p) = \max\{M_1(\alpha,p), M_2(\alpha,p)\}$,

$$M_1(\alpha,p) = 2.(n(\alpha,p))^{\alpha} + (\pi.(n(\alpha,p))^2)^{\frac{-1}{p}},$$
$$M_2(\alpha,p) = (2.\sqrt[p]{4}).(\sqrt[p]{4} - 1)^{-1}.(n(\alpha,p))^{\alpha},$$
$$n(\alpha,p) = (\alpha.p.\sqrt[p]{\pi})^{\frac{-p}{2+\alpha.p}}$$

Lemma 2.3.
For $D_{0R} = \{(z,\varphi) \in D_0 : z \in \overline{G}, |\varphi| \leq R\}, R > 0$, $B_\alpha(0;R) = \{\varphi \in H_\alpha(\overline{G}) : \|\varphi\| \leq R\}$, $\alpha \in (0,1)$ *and* $\varphi \in B_\alpha(0;R)$, $f, g : D_{0R} \to \mathbb{C}$ *let* $f_1(z) = f(z,\varphi(z))$, $g_1(z) = g(z,\varphi(z))$, $z \in \overline{G}$. *Then for the functions* $f_1, g_1 : \overline{G} \to \mathbb{C}$ *the following inequalities are true*

$$|f_1(z)| \leq m_0 + m_2.R, \; z \in \overline{G},$$
$$|f_1(z_1) - f_1(z_2)| \leq (m_1 + m_2.R).|z_1 - z_2|^{\alpha}, \qquad (7)$$
$$z_1, z_2 \in \overline{G},$$
$$|g_1(z)| \leq n_0 + n_2.R, \; z \in \overline{G},$$
$$|g_1(z_1) - g_2(z_2)| \leq (n_1 + n_2.R).|z_1 - z_2|^{\alpha}, \qquad (8)$$
$$z_1, z_2 \in \overline{G}.$$

Here,

$$m_0 = \max\{|f(z,0)| : z \in \overline{G}\},$$
$$n_0 = \max\{|g(z,0)| : z \in \overline{G}\}.$$

The proof of the Lemma 2.3 is obvious from the Inequalities (2), (3) and the assumptions of the lemma.

Corollary 2.1. If the assumptions of the Lemma 2.3 are satisfied then for, $f_1, g_1 \in H_\alpha(\overline{G})$, $\alpha \in (0,1)$ *we have*

$$\|f_1\|_\alpha \le m_0 + m_1 + 2Rm_2,$$
$$\|g_1\|_\alpha \le n_0 + n_1 + 2Rn_2. \tag{9}$$

Now, for $f \in H_{\alpha,1}(m_1, m_2; D_0)$, $g \in H_{\alpha,1}(n_1, n_2; D_0)$ and $\varphi \in H_\alpha(\overline{G})$, $\alpha \in (0,1)$ let us define the functions $\widetilde{f}_1, \widetilde{g}_1 : \overline{G} \to \mathbb{C}$ as given below

$$\widetilde{f}_1(z) = T_G f(., \varphi(.))(z),$$
$$\widetilde{g}_1(z) = \Pi_G g(., \varphi(.))(z). \tag{10}$$

For the bounded operators $T_G, \Pi_G : H_\alpha(\overline{G}) \to H_\alpha(\overline{G})$, $\alpha \in (0,1)$ let us define the following norms:

$$\|T_G\|_\alpha \equiv \sup\{\|T_G\varphi\|_\alpha : \varphi \in H_\alpha(\overline{G}),\ \|\varphi\|_\alpha \le 1\} \tag{11}$$

$$\|\Pi_G\|_\alpha \equiv \sup\{\|\Pi_G\varphi\|_\alpha : \varphi \in H_\alpha(\overline{G}),\ \|\varphi\|_\alpha \le 1\} \tag{12}$$

[11], [14].

Lemma 2.4. If $f \in H_{\alpha,1}(m_1, m_2; D_0)$, $g \in H_{\alpha,1}(n_1, n_2; D_0)$ *and* $\varphi \in B_\alpha(0;R)$, $\alpha \in (0,1)$ *then in this case* $\widetilde{f}_1, \widetilde{g}_1 \in H_\alpha(\overline{G})$ *and we have*

$$\|\widetilde{f}_1\|_\alpha \le L_1, \quad \|\widetilde{g}_1\|_\alpha \le L_2. \tag{13}$$

Here,

$$L_1 = (m_0 + m_1 + 2m_2.R).\|T_G\|_\alpha,$$
$$L_2 = (n_0 + n_1 + 2n_2.R).\|\Pi_G\|_\alpha.$$

The proof of the Lemma 2.4 is obvious from the definitions of the functions $\widetilde{f}_1, \widetilde{g}_1 : \overline{G} \to \mathbb{C}$ and corollary 2.1.

Lemma 2.5. Let $f \in H_{\alpha,1}(m_1, m_2; D_0)$, $g \in H_{\alpha,1}(n_1, n_2; D_0)$, $F \in H_{\alpha,1,1,1}(l_1, l_2, l_3, l_4; D_R)$ *and* $D_R = \{(z, \varphi, t, s) \in D_0 : z \in \overline{G}, |\varphi| \le R, |t| \le R, |s| \le R\}$. *Then for the function* $\mathcal{F} : \overline{G} \to \mathbb{C}$ *that is defined as* $\mathcal{F}(z) = F(z, \varphi(z), (T_G \circ f_1)(z), (\Pi_G \circ f_1)(z))$, $z \in \overline{G}$ *and for* $\varphi \in B_\alpha(0;R)$, $\alpha \in (0,1)$ *we have*

$$|\mathcal{F}(z)| \le l_0 + l_2.R + l_3.L_1 + l_4.L_2,\ z \in \overline{G},$$
$$|\mathcal{F}(z_1) - \mathcal{F}(z_2)| \le (l_1 + l_2.R + l_3.L_1 + l_4.L_2)\times$$
$$|z_1 - z_2|^\alpha,\ z_1, z_2 \in \overline{G}.$$

Here, $l_0 = \|F(.,0,0,0)\|_\infty$.

The proof of the Lemma 2.5 is obvious from the definition of the function $\mathcal{F} : \overline{G} \to \mathbb{C}$, the Inequality (13) and from the assumption of lemma.

Corollary 2.2. If the assumptions of the Lemma 2.5 are satisfied then for $\mathcal{F} \in H_\alpha(\overline{G})$ *we have*

$$\|\mathcal{F}\|_\alpha \le L = 2\{\max(l_0, l_1) + l_2.R + l_3.L_1 + l_4.L_2\}.$$

Corollary 2.3. If the assumptions of the Lemma 2.5 are satisfied then the operator A that is defined with

$$A(\varphi)(z) =$$
$$\lambda.F(z, \varphi(z), (T_G \circ f_1)(z), (\Pi_G \circ g_1)(z), z \in \overline{G} \tag{14}$$

transforms the sphere $B_\alpha(0;R)$ *to the sphere* $B_\alpha(0;|\lambda|L)$.

Corollary 2.4. If the assumptions of the Lemma 2.5 are satisfied and if $|\lambda|L \le R$ *then the operator A that is defined with* 14) *transforms the sphere* $B_\alpha(0;R)$ *to itself.*

Lemma 2.6. The sphere $B_\alpha(0;R)$, $\alpha \in (0,1)$ *is a compact set of the space* $(H_\alpha(\overline{G}); \|.\|_\infty)$.

Proof. From the definition of the sphere $B_\alpha(0;R)$, $\alpha \in (0,1)$ for $\varphi \in B_\alpha(0;R)$ we have $\|\varphi\|_\infty \le R$, therefore, it is clear that the sphere $B_\alpha(0;R)$ is uniform bounded in the space $(H_\alpha(\overline{G}); \|.\|_\infty)$. Furthermore, for every $\varepsilon > 0$ if we take $\delta = (\varepsilon / R)^{1/\alpha}$ then for every $z_1, z_2 \in \overline{G}$ and $\varphi \in B_\alpha(0;R)$ when $|z_1 - z_2| < \delta$ we have $|\varphi(z_1) - \varphi(z_2)| \le R.|z_1 - z_2|^\alpha < \varepsilon$. From here we can see that the elements of the sphere $B_\alpha(0;R)$ are continuous at same order. Thus, as required by the Arzela-Ascoli theorem about compactness the sphere $B_\alpha(0;R)$ is a compact set of the space $(H_\alpha(\overline{G}); \|.\|_\infty)$.

Corollary 2.5. The sphere $B_\alpha(0;R)$, $\alpha \in (0,1)$ *is a complete subspace of the space* $(H_\alpha(\overline{G}); \|.\|_\infty)$.

Now, let us define the following two transformations, which are in the space $(H_\alpha(\overline{G});\|.\|_\infty)$. For $\varphi, \widetilde{\varphi} \in H_\alpha(\overline{G})$, $\alpha \in (0,1)$, $p > 1$ let $d_\infty(\varphi, \widetilde{\varphi}) = \|\varphi - \widetilde{\varphi}\|_\infty$, $d_p(\varphi, \widetilde{\varphi}) = \|\varphi - \widetilde{\varphi}\|_p$. Then it is easy to show that the transformations $d_\infty : H_\alpha(\overline{G}) \to [0,+\infty)$, $d_p : H_\alpha(\overline{G}) \to [0,+\infty)$ are both defines a metric on the space $H_\alpha(\overline{G})$, $\alpha \in (0,1)$, consequently, it can be easily seen that $(H_\alpha(\overline{G});\|.\|_\infty)$ and $(H_\alpha(\overline{G});\|.\|_p)$ are metric spaces.

Lemma 2.7.. The convergence is equivalent for the metrics d_∞ and d_p in the subspace $B_\alpha(0;R)$, $\alpha \in (0,1)$, $p > 1$.
Proof of Lemma 2.7 is direct result of (6) and

$$d_p(\varphi_0, \varphi_n) \le \left[\pi.(d/2)^2\right]^{1/p}.d_\infty(\varphi_0, \varphi_n)$$

inequalities, for all $\varphi_0, \varphi_n \in B_\alpha(0;R)$, $n = 1,2,...$ $\alpha \in (0,1)$.

Now, for bounded operators $T_G, \Pi_G : L_p(\overline{G}) \to L_p(\overline{G})$, $p > 1$ let us define the following norms

$$\|T_G\|_{L_p(\overline{G})} \equiv \sup\{\|T_G\varphi\|_p : \|\varphi\|_p \le 1\},$$

$$\|\Pi_G\|_{L_p(\overline{G})} \equiv \sup\{\|\Pi_G\varphi\|_p : \|\varphi\|_p \le 1\}$$

[11], [14].

Lemma 2.8. Let $f \in H_{\alpha,1}(m_1, m_2; D_{0R})$, $g \in H_{\alpha,1}(n_1, n_2; D_{0R})$ and $F \in H_{\alpha,1,1,1}(l_1, l_2, l_3, l_4; D_R)$, $\alpha \in (0,1)$, $p > 1$. In this case, for $\varphi, \widetilde{\varphi} \in B_\alpha(0;R)$ the operator A that is defined with the Equality (14) satisfies the following inequality

$$d_p(A(\varphi), A(\widetilde{\varphi})) \le |\lambda| M_3(p).d_\infty(\varphi, \widetilde{\varphi}). \quad (15)$$

Here,

$$M_3(p) = \left(l_2 + m_2 l_3.\|T_G\|_{L_p(\overline{G})} + n_2 l_4.\|\Pi_G\|_{L_p(\overline{G})}\right)\times (m(G))^{1/p}.$$

Proof. For $z \in \overline{G}$ and $\varphi, \widetilde{\varphi} \in B_\alpha(0;R)$ from the assumption of the lemma and definition of the operator A we have

$$|A(\varphi)(z) - A(\widetilde{\varphi})(z)| = |\lambda| \times \left| \begin{matrix} F(z, \varphi(z), T_G f(., \varphi(.))(z), \Pi_G g(., \varphi(.))(z)) - \\ F(z, \widetilde{\varphi}(z), T_G f(., \widetilde{\varphi}(.))(z), \Pi_G g(., \widetilde{\varphi}(.))(z)) \end{matrix} \right|$$

$$\le |\lambda|. \begin{bmatrix} l_2|\varphi(z) - \widetilde{\varphi}(z)| + \\ l_3|T_G(f(., \varphi(.)) - f(., \widetilde{\varphi}(.)))(z)| + \\ l_4|\Pi_G(g(., \varphi(.)) - g(., \widetilde{\varphi}(.)))(z)| \end{bmatrix}.$$

As for this, from the assumptions on the functions f, g and F and in accordance with Minkowski inequality we have,

$$\left(\iint_G |A(\varphi)(z) - A(\widetilde{\varphi})(z)|^p dxdy\right)^{1/p} \le |\lambda| \times$$

$$\left(\iint_G \begin{bmatrix} l_2|\varphi(z) - \widetilde{\varphi}(z)| + \\ l_3|T_G(f(., \varphi(.)) - f(., \widetilde{\varphi}(.)))(z)| + \\ l_4|\Pi_G(g(., \varphi(.)) - g(., \widetilde{\varphi}(.)))(z)| \end{bmatrix}^p dxdy\right)^{1/p}$$

$$\le |\lambda|. \begin{pmatrix} l_2\|\varphi - \widetilde{\varphi}\|_p + \\ l_3.\|T_G\|_{L_p(\overline{G})}.\|f(., \varphi(.)) - f(., \widetilde{\varphi}(.))\|_p + \\ l_4.\|\Pi_G\|_{L_p(\overline{G})}.\|g(., \varphi(.)) - g(., \widetilde{\varphi}(.))\|_p \end{pmatrix}$$

$$\le |\lambda|. \begin{pmatrix} l_2 + m_2 l_3.\|T_G\|_{L_p(\overline{G})} + \\ n_2 l_4.\|\Pi_G\|_{L_p(\overline{G})} \end{pmatrix}.\|\varphi - \widetilde{\varphi}\|_p$$

and therefore we obtain

$$\left(\iint_G |A(\varphi)(z) - A(\widetilde{\varphi})(z)|^p dxdy\right)^{1/p} \le$$

$$|\lambda|.\left(l_2 + m_2 l_3.\|T_G\|_{L_p(\overline{G})} + n_2 l_4.\|\Pi_G\|_{L_p(\overline{G})}\right)\|\varphi - \widetilde{\varphi}\|_p.$$

From this inequality, it is seen that the Inequality (15) is true. With this, the lemma is proved.

Lemma 2.9. If the assumptions of the Lemma 2.8 are satisfied and $|\lambda|.L \le R.$ Then the operator $A : B_\alpha(0;R) \to B_\alpha(0;R)$, $\alpha \in (0,1)$ is a continuous operator for the metric d_∞.

Proof. Let $\varphi_0, \varphi_n \in B_\alpha(0;R)$, $n = 1,2,...$ $\alpha \in (0,1)$ and $\lim_{n\to\infty} d_\infty(\varphi_n, \varphi_0) = 0$. We want to show that $\lim_{n\to\infty} d_\infty(A(\varphi_n), A(\varphi_0)) = 0$. From the Inequality (6) we can write

$$\|A(\varphi_n) - A(\varphi_0)\|_\infty \le$$

$$M(\alpha, p)\|A(\varphi_n) - A(\varphi_0)\|_\alpha^{\frac{2}{2+\alpha p}} \times.$$

$$\|A(\varphi_n) - A(\varphi_0)\|_p^{\frac{\alpha p}{2+\alpha p}}$$

Thus, due to the Inequality (15) we will have

$$\|A(\varphi_n)-A(\varphi_0)\|_\infty \le M(\alpha,p)(2R)^{\frac{2}{2+\alpha p}} \times$$

$$(|\lambda|.M_3(p))^{\frac{\alpha p}{2+\alpha p}}.(d_\infty(\varphi_n,\varphi_0))^{\frac{\alpha p}{2+\alpha p}}.$$

From this inequality, it can easily be seen that if $\lim_{n\to\infty} d_\infty(\varphi_n,\varphi_0)=0$ then

$$\lim_{n\to\infty} d_\infty(A(\varphi_n),A(\varphi_0))=0.$$

III. MAIN RESULTS

Now, we can give the theorem about the existence of the solution of the Equation (1).

Theorem 3.1. If $f\in H_{\alpha,1}(m_1,m_2;D_{0R})$, $g\in H_{\alpha,1}(n_1,n_2;D_{0R})$ *and* $F\in H_{\alpha,1,1,1}(l_1,l_2,l_3,l_4;D_R)$, $\alpha\in(0,1)$ *and* $|\lambda|L\le R$ *then the nonlinear singular integral equation* (1) *has at least one solution in the sphere* $B_\alpha(0;R)$.

Proof. If $|\lambda|L\le R$ then the operator A which is defined with the Formula (14) transforms the convex, closed and compact set $B_\alpha(0;R)$ to itself in the Banach space $(H_\alpha(\overline{G});\|.\|_\alpha)$, $\alpha\in(0,1)$. Thus, the operator A is a compact operator in the space $H_\alpha(\overline{G})$. In addition to this, since the operator A is also continuous on the set $B_\alpha(0;R)$, it is completely continuous operator. In that case, due to the Schauder fixed-point principle the operator A has a fixed point in the set $B_\alpha(0;R)$. Consequently, the Equation (1) has a solution in the space $H_\alpha(\overline{G})$, $\alpha\in(0,1)$.
With this, the theorem is proved.

Now, we will investigate the uniqueness of the solution of the Equation (1) and the problem that how can we find the approximate solution. For this, we will use a more useful modified version of the Banach fixed-point principle for the uniqueness of the solution of operator equations [16].

Theorem 3.2. Suppose the following assumptions are satisfied:

i. (X,ρ_1) *is a compact metric space;*

ii. In the space X, every sequence that is convergent for the metric ρ_1 *is also convergent for a second metric* ρ_2 *that is defined on X;*

iii. The operator $A:X\to X$ *is a contraction mapping for the metric* ρ_2, *that is, for every* $x,y\in X$ *there exist a number* $q\in[0,1)$ *such that*

$$\rho_2(Ax,Ay)\le q.\rho_2(x,y).$$

In this case, the operator equation $x=Ax$ *has only unique solution* $x_*\in X$ *and for* $x_0\in X$ *to be any initial approximation, the velocity of the convergence of the sequence* (x_n) *to the solution* x_* *is determined by the following inequality*

$$\rho_2(x_n,x_*)\le q^n.(1-q)^{-1}.\rho_2(x_1,x_0).$$

Here, the terms of the sequence are defined by $x_n = Ax_{n-1}$, $n=1,2,\dots$.

On the basis of this theorem and preceding information, we can give the following theorem about the uniqueness of the solution of the Equation (1) and also about finding the solution.

Theorem 3.3. If $f\in H_{\alpha,1}(m_1,m_2;D_{0R})$, $g\in H_{\alpha,1}(n_1,n_2;D_{0R})$, $F\in H_{\alpha,1,1,1}(l_1,l_2,l_3,l_4;D_R)$, $\alpha\in(0,1)$, $|\lambda|L\le R$, $l=|\lambda|.(l_2+m_2l_3.\|T_G\|_{L_p(\overline{G})}+ n_2l_4.\|\Pi_G\|_{L_p(\overline{G})})<1$, $p>1$ *then the nonlinear singular integral equation* (1) *has only unique solution* $\varphi_*\in B_\alpha(0;R)$ *and for* $\varphi_0\in B_\alpha(0;R)$ *to be any initial approximation, this solution can be found as a limit of the sequence* (φ_n), $n=1,2,\dots$ *whose terms are defined as below*

$$\varphi_n(z)=\lambda.F\begin{pmatrix} z,\varphi_{n-1}(z),T_Gf(.,\varphi_{n-1}(.))(z), \\ \Pi_Gg(.,\varphi_{n-1}(.))(z)\end{pmatrix},$$
$$z\in\overline{G},\ n=1,2,\dots. \tag{16}$$

Furthermore, for the velocity of the convergence the following evaluation is right,

$$d_p(\varphi_n,\varphi_*)\le l^n.(1-l)^{-1}.d_p(\varphi_1,\varphi_0),\ n=1,2,\dots. \tag{17}$$

Proof. In order to prove the first part of the theorem it is sufficient to show that the assumptions i-iii of the Theorem 3.2 is satisfied. In accordance with Lemma 2.6 and Corollary 2.5 for $\varphi,\widetilde{\varphi}\in B_\alpha(0;R)$ and $d_p(\varphi,\widetilde{\varphi}) =_\infty \|\varphi-\widetilde{\varphi}\|_\infty$, $(B_\alpha(0;R);d_\infty)$ is a compact metric space. Therefore, for $X=B_\alpha(0;R)$ and $\rho_1=d_\infty$ the assumption ***i.*** of the Theorem 3.2 is satisfied.

If we take $\rho_2=d_p$, $\mathrm{p}>1$, from the Lemma 2.7 it is obvious that the assumption *ii.* of the Theorem 3.2 is satisfied.

Now, let us show that when $l < 1$ then the operator A is a contraction transformation with respect to the d_p metric and so we have shown that the assumption iii. of the Theorem 3.2 is satisfied.

For any $\varphi_1, \varphi_2 \in B_\alpha(0;R)$, same as the proof of the Lemma 2.8 the following inequality can be proven

$$\|A(\varphi_1) - A(\varphi_2)\|_p \le l.\|\varphi_1 - \varphi_2\|_p .$$

Here, $l = |\lambda|.(l_2 + m_2 l_3.\|T_G\|_{L_p(\overline{G})} + n_2 l_4.\|\Pi_G\|_{L_p(\overline{G}}).$

Therefore, for $\varphi_1, \varphi_2 \in B_\alpha(0;R)$ we can write

$$d_p(A(\varphi_1), A(\varphi_2)) \le l.d_p(\varphi_1, \varphi_2) . \quad (18)$$

Thus, when $l < 1$ it can be seen from the Inequality (18) that the operator A is a contraction mapping with respect to the d_p metric that is defined on the $B_\alpha(0;R)$.

Thus, from the assumptions of the theorem; *i.* $B_\alpha(0;R)$ is a compact metric space; *ii.* In the space $B_\alpha(0;R)$, every sequence that is convergent for the metric d_∞ is also convergent for the metric d_p that is defined on $B_\alpha(0;R)$ and *iii.* $A : B_\alpha(0;R) \to B_\alpha(0;R)$ is a contraction mapping with respect to the norm d_p.

Therefore, because of the Theorem 3.2 the Equation (1) has only unique solution $\varphi_* \in B_\alpha(0;R)$.

Now, let us show that for $\varphi_0 \in B_\alpha(0;R)$ to be an initial approximation point, this solution is the limit of the sequence (φ_n), $n = 1,2,...$ whose terms are defined by (16) or by $\varphi_n(z) = A(\varphi_{n-1}(z))$, $z \in \overline{G}$, $n = 1,2,...$.

For every $n = 1,2,...$ from the Inequality (18) we have

$$d_p(\varphi_{n+1}, \varphi_n) = d_p(A(\varphi_n), A(\varphi_{n-1})) \le l.d_p(\varphi_n, \varphi_{n-1})$$

so from here we can write

$$d_p(\varphi_{n+1}, \varphi_n) \le l.d_p(\varphi_n, \varphi_{n-1})$$

and from this inequality , we will have

$$d_p(\varphi_{n+1}, \varphi_n) \le l^n.d_p(\varphi_1, \varphi_0) .$$

So, for any natural numbers *m* and *n* we can write

$$d_p(\varphi_{n+m}, \varphi_n) \le l^n.d_p(\varphi_m, \varphi_0).$$

Furthermore, because

$$d_p(\varphi_m, \varphi_0) \le (l^{m-1} + l^{m-2} + ... + l + 1) d_p(\varphi_1, \varphi_0)$$

we will have,

$$d_p(\varphi_{n+m}, \varphi_n) \le (1 - l^m).(1 - l)^{-1}.l^n.d_p(\varphi_1, \varphi_0). \quad (19)$$

From the Inequality (19), it can be seen that the sequence (φ_n), $n = 1,2,...$ is a Cauchy sequence with respect to the metric d_p. Therefore, since $(B_\alpha(0;R); d_p)$ is a complete metric space there exist the limit $\varphi_* \in B_\alpha(0;R)$ such that $\lim_{n\to\infty} \varphi_n = \varphi_*$ or $\lim_{n\to\infty} d_p(\varphi_n, \varphi_*) = 0.$

From (18), for every natural number *n* since

$$d_p(\varphi_{n+1}, A(\varphi_*)) = d_p(A(\varphi_n), A(\varphi_*)) \le l.d_p(\varphi_n, \varphi_*) ,$$

we have

$$\lim_{n\to\infty} d_p(\varphi_{n+1}, A(\varphi_*)) = 0 ,$$

and consequently, $\lim_{n\to\infty} \varphi_{n+1} = A(\varphi_*)$ in other words $\varphi_* = A(\varphi_*)$. With this, it can be seen that the operator equation $\varphi(z) = A(\varphi(z))$, $z \in \overline{G}$, consequently, the solution of the Equation (1) is the limit of the sequence whose terms are defined by $\varphi_n(z) = A(\varphi_{n-1}(z))$, $z \in \overline{G}$, $n = 1,2,...$. Furthermore, as because $\lim_{m\to\infty} \varphi_{n+m}(z) = \varphi_*(z)$ and $\lim_{m\to\infty} l^m = 0$ from the Inequality (19) the convergence velocity of the sequence $(\varphi_n(z))$, $z \in \overline{G}$, $n = 1,2,...$ to the solution $\varphi_*(z)$ is given by the Inequality (17).

From Theorem 3.3 and Lemma 2.7 we will have the following result.

Corollary 3.1. In the space $B_\alpha(0;R)$, $\alpha \in (0,1)$ *the sequence* $(\varphi_n(z))$, $z \in \overline{G}$, $n = 1,2,...$ *whose terms are defined by* (16) (*or defined by* $\varphi_n(z) = A(\varphi_{n-1}(z))$, $z \in \overline{G}$, $n = 1,2,...$) *also converges to the unique solution of the Equation* (1) *with respect to the metric* d_∞.

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