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Analysis of the Coupled Stretching–Bending Problem of Stiffened Plates by a BEM Formulation Based on Reissner's Hypothesis

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Abstract**—**In this work, the plate bending formulation of the boundary element method - BEM, based on the Reissner's hypothesis, is extended to the analysis of plates reinforced by beams taking into account the membrane effects. The formulation is derived by assuming a zoned body where each sub-region defines a beam or a slab and all of them are represented by a chosen reference surface. Equilibrium and compatibility conditions are automatically imposed by the integral equations, which treat this composed structure as a single body. In order to reduce the number of degrees of freedom, the problem values defined on the interfaces are written in terms of their values on the beam axis. Initially are derived separated equations for the bending and stretching problems, but in the final system of equations the two problems are coupled and can not be treated separately. Finally are presented some numerical examples whose analytical results are known to show the accuracy of the proposed model.

Keywords**—**Boundary elements, Building floor structures, Plate bending.

I. Introduction

THE boundary element method (BEM) has already proved to be a suitable numerical tool to deal with plate bending problems. The method is particularly recommended to evaluate internal force concentrations due to loads distributed over small regions that very often appear in practical problems. Moreover, the same order of errors is expected when computing deflections, slopes, moments and shear forces. Shear forces, for instance, are much better evaluated when compared with other numerical methods. They are not obtained by differentiating approximation function as for other numerical techniques.

Using BEM coupled with the finite element method (FEM) is the natural numerical procedure to analyze plate reinforced by beams, where the BEM is used to represent the plate elements and the FEM to approximate the beam elements. Regarding this numerical technique several formulations have already been proposed ([1]-[3]), where the BEM formulation is based either on Kirchhoff's or Reissner's hypothesis. However, for complex floor structures the number of degrees of freedom may increase rapidly diminishing the solution accuracy.

An alternative scheme to reduce the number of degrees of freedom has been recently proposed by FERNANDES & VENTURINI in [4] and [5] using only a BEM formulation based on Kirchhoff's hypothesis. In both work the building floor is modeled by a zoned plate being each sub-region the representation of either a beam or a slab. This composed structure is treated as a single body, being the equilibrium and compatibility conditions automatically taken into account. In the first work is proposed a formulation to perform simple bending analysis where the tractions are eliminated along the interfaces. Moreover in order to reduce the number of degrees of freedom some Kinematic assumptions were made along the beam width. In the second work this formulation is extended to take into account the membrane effects which are associated with bending due to the relative positions of the structural elements. In this case the in-plane tractions can not be eliminated on the interfaces, so that they should also be approximated along the beam width. For the plate stretching-bending formulation, the resulting integral equations of both, the bending and the plane stress problems, are coupled and cannot be treated separately.

In this work the BEM formulation developed in [5] is modified to take into account the Reissner's hypothesis instead of the Kirchhoff's. In the proposed model the tractions related to the bending problem is no longer eliminated on the interfaces. Therefore traction and displacements related to both problems (bending and stretching) are approximated along the beam width, leading to a model where the problem values are defined only on the beams axis and on the plate boundary without beams. The accuracy of the proposed model is illustrated by numerical examples whose analytical results are known.

Note that in the classical theory (Kirchhoff's) [5], [6] and [7] are defined only four boundary values and its inaccuracy turns out to be important for thick plates, especially in the edge zone of the plate and around holes whose diameter is not larger than the plate thickness. In the Reissner's theory ([8], [9] and [10]) which can be used either for thin or thick plates,

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are defined six boundary values and it is more accurate because it takes into account the shear deformation effect.

II. Basic Equations

Without loss of generality, let us consider the plate depicted in Fig. 1, where h_1, h_2 and h_3 are the thicknesses of the sub-regions Ω_1, Ω_2 and Ω_3, whose external boundaries are Γ_1, Γ_2 and Γ_3, respectively. The total external boundary is given by Γ while Γ_{jk} represents the interface between the adjacent sub-regions Ω_j and Ω_k. The Cartesian system of co-ordinates (axes x_1, x_2 and x_3) is defined on a reference surface, whose distance to the sub-regions middle surfaces are given by c_1, c_2 and c_3 .

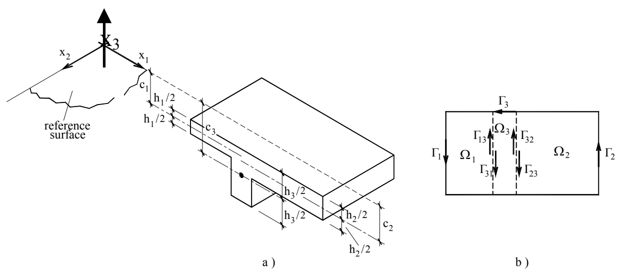

Fig. 1 (a) General zoned plate domain; (b) reference surface view

Let us consider initially, the bending problem. For a point placed at any of those plate sub-regions, the following equations are defined:

-The equilibrium equations in terms of internal forces:

$$\mathrm{M}_{ij},_j - Q_i = 0 \qquad \text{i, j} = 1, 2 \tag{1}$$

$$Q_i,_i + g = 0 \tag{2}$$

where g is the distributed load acting on the plate middle surface, m_{ij} are bending and twisting moments and Q_i represents shear forces.

-The generalised internal forces written in terms of displacement:

$$M_{ij} = \frac{D(1-\nu)}{2}\left(\phi_i,_j + \phi_j,_i + \frac{2\nu}{1-\nu}\phi_k,_k\,\delta_{ij}\right) + \frac{\nu g}{(1-\nu)\lambda^2}\delta_{ij} \tag{3}$$

$$Q_i = \frac{D(1-\nu)}{2}\lambda^2\left(\phi_i + w,_i\right) \tag{4}$$

where $i, j, k = 1, 2$; $_{\mathrm{i}}$ is the rotation in the i direction, w the deflection, $\phi_k,_l$ the plate curvature, $\psi_{3l} = \phi_l + w,_l$ the shear deformation, $D = Eh^3/(1-\nu^2)$ the flexural rigidity, ν the Poisson's ration, λ a constant related to shear effect given by $\lambda = \sqrt{10}/h$ and δ_{ij} is the Kronecker delta.

-Finally, the plate bending differential equations given by:

$$Q_i - \frac{1}{\lambda^2}\nabla^2 Q_i + \frac{1}{(1-\nu)\lambda^2}\frac{\partial g}{\partial x_i} = -D\frac{\partial}{\partial x_i}\nabla^2 w \qquad \text{i=1, 2} \tag{5}$$

$$\nabla^4 w = \frac{1}{D}\left[g - \frac{(2-\nu)}{(1-\nu)\lambda^2}\nabla^2 g\right] \tag{6}$$

where $w,_{iijj} = \nabla^4 w$, being ∇^4 the bi-harmonic operator; $w,_{ii} = \nabla^2 w$ being ∇^2 the bi-Laplacian operator.

Equations (5) and (6) result into the set of differential equations, being (5) and. (6) a second and fourth order equation, respectively, leading therefore to six independent boundary values: M_n; M_{ns}, Q_n, w, ϕ_n and ϕ_s, being (n, s) the local co-ordinate system, with n and s referred to the plate boundary normal and tangential directions, respectively.

Considering now the stretching problem, the in-plane equilibrium equation is:

$$N_{ij},_j + b_i = 0 \tag{7}$$

where b_i represents the body forces distributed over the plate middle surface and N_{ij} is the membrane internal force, which, assuming plane stress conditions, is given by:

$$N_{ij} = Gh\left[\frac{2\nu}{(1-\nu)}u_k,_k\,\delta_{ij} + (u_i,_j + u_j,_i)\right] \tag{8}$$

The in-plane equilibrium can also be written in terms of displacements by replacing (8) into (7) as follows:

$$\frac{1+\nu}{1-\nu}u_j,_{ij} + u_i,_{jj} + b_i/Gh = 0 \tag{9}$$

The problem definition is then completed by assuming the following boundary conditions over Γ: $U_i = \overline{U}_i$ on Γ_u (generalised displacements: deflections, rotations and in-plane displacements) and $P_i = \overline{P}_i$ on Γ_p (generalised tractions: bending and twisting moments, shear forces and in-plane tractions), where $\Gamma_u \cup \Gamma_p = \Gamma$.

III. Integral Representations

Let us initially consider the simple bending problem. Considering the plate equilibrium equations (1) and (2) the following weighted residual equation can be obtained for a simple plate:

$$\int_{\Omega}\left[\phi^*_{ki}\left(M_{ij},_j - Q_i\right) + \left(Q_i,_i + g\right)w^*_k\right]d\Omega = \int_{\Gamma_u}\left[\left(\overline{\phi}_i - \phi_i\right)M^*_{ki} + \left(\overline{w} - w\right)Q^*_{kn}\right]d\Gamma - \int_{\Gamma_P}\left[\left(\overline{M}_i - M_i\right)\phi^*_{ki} + \left(\overline{Q}_n - Q_n\right)w^*_k\right]d\Gamma \tag{10}$$

where i, j =1, 2 and k=1, 2, 3; k = 1, 2 refers to unit moments applied in the x_1 and x_2 directions and k=3 refers to a unit load acting in the x_3 direction.

Integrating (10) by parts twice, considering (3) and (4) and writing the values in terms of the local system of coordinates (n,s), the following integral equation of the generalised displacements can be obtained:

$$c(q)U_k(q)=\int_{\Omega_g} g\left[w_k^*(q,p)-\frac{\nu}{(1-\nu)\lambda^2}\phi_{ki}^*{}_{,i}(q,p)\right]d\Omega+$$
$$-\int_{\Gamma}\left[\phi_n(P)M_{kn}^*(q,P)+\phi_s(P)M_{kns}^*(q,P)+w(P)Q_{kn}^*(q,P)\right]d\Gamma+$$
$$+\int_{\Gamma}\left[M_n(P)\phi_{kn}^*(q,P)+M_{ns}(P)\phi_{ks}^*(q,P)+Q_n(P)w_k^*(q,P)\right]d\Gamma \quad (11)$$

where k=m, l, 3; i=1, 2; Ω_g is the area where the load g is distributed, the free term value $c(q)$ depends on the position of the point q: $c(q)=0$ for external points $c(q)=1$ for internal points and $c(Q)=1/2$ for boundary points; $U_m=\phi_m$, $U_l=\phi_l$ and $U_3=w$, being m and l the local system (n, s) for boundary points or any direction for internal points.

Let us now consider a zoned plate as the one depicted in the Fig. 1, as example. In this case (11) is valid to each sub-region separately. Then, taking into account the equilibrium and compatibility conditions, writing (11) to all sub-regions and summing them the following integral equation for the zoned plate can be obtained:

$$U_k(q)=\sum_{j=1}^{N_s}\int_{\Omega_g} g\left[w_k^{*j}(q,p)-\frac{\nu}{(1-\nu)\lambda^2}\phi_{ki}^{*j}{}_{,i}(q,p)\right]d\Omega+$$
$$-\sum_{j=1}^{N_s}\int_{\Gamma_1}\left[\phi_n(P)M_{kn}^{*j}(q,P)+\phi_s(P)M_{kns}^{*j}(q,P)+w(P)Q_{kn}^{*j}(q,P)\right]d\Gamma+$$
$$-\sum_{j=1}^{N_{\text{int}}}\int_{\Gamma_{ja}}\left\{\phi_n(P)\left[M_{kn}^{*j}(q,P)-M_{kn}^{*a}(q,P)\right]+\phi_s(P)\left[M_{kns}^{*j}(q,P)-M_{kns}^{*a}(q,P)\right]+\right.$$
$$\left.+w(P)\left[Q_{kn}^{*j}(q,P)-Q_{kn}^{*a}(q,P)\right]\right\}d\Gamma+$$
$$+\sum_{j=1}^{N_s}\int_{\Gamma_1}\left[M_n(P)\phi_{kn}^{*j}(q,P)+M_{ns}(P)\phi_{ks}^{*j}(q,P)+Q_n(P)w_k^{*j}(q,P)\right]d\Gamma$$
$$+\sum_{j=1}^{N_{\text{int}}}\int_{\Gamma_{ja}}\left\{M_n(P)\left[\phi_{kn}^{*j}(q,P)-\phi_{kn}^{*a}(q,P)\right]+M_{ns}(P)\left[\phi_{ks}^{*j}(q,P)-\phi_{ks}^{*a}(q,P)\right]+\right.$$
$$\left.+Q_n(P)\left[w_k^{*1}(q,P)-w_k^{*a}(q,P)\right]\right\}d\Gamma \quad (12)$$

where N_s is the sub-regions number and N_{int} the interfaces number, Γ_{ja} represents a interface for which the subscript a denotes the adjacent sub-region to Ω_j

Note that in both integrals along the interface Γ_{ja} all values are related to the local system defined on Γ_{ja} and the generalised fundamental values U_{ki}^{*a} and P_{ki}^{*a} are given in terms of the rigidity D and thickness h of the sub-region Ω_a. In this case the tractions can not be eliminated along the interfaces, because it is not possible to write the fundamental expressions related to the sub-region Ω_j in terms of their values in a chosen sub-region, as it has been considered for the formulation based on Kirchhoff's hypothesis [5].

In (12) all sub-regions are still referred to their middle surface. The bending equation for the coupled stretching-bending problem is obtained by writing the moment values on the middle surface of sub-region Ω_j in terms of their values on the reference surface (M_n^r and M_{ns}^r), as follow:

$$M_n^j=M_n^r+p_nc_j \quad (13)$$
$$M_{ns}^j=M_{ns}^r+p_sc_j \quad (14)$$

where p_n and p_s are the in-plane tractions.

Replacing (13) and (14) into (12) the integral equation for the bending problem is finally obtained:

$$U_k(q)=\sum_{j=1}^{N_s}\int_{\Omega_g} g\left[w_k^{*j}(q,p)-\frac{\nu}{(1-\nu)\lambda^2}\phi_{ki}^{*j}{}_{,i}(q,p)\right]d\Omega+$$
$$-\sum_{j=1}^{N_s}\int_{\Gamma_1}\left[\phi_n(P)M_{kn}^{*j}(q,P)+\phi_s(P)M_{kns}^{*j}(q,P)+w(P)Q_{kn}^{*j}(q,P)\right]d\Gamma+$$
$$-\sum_{j=1}^{N_{\text{int}}}\int_{\Gamma_{ja}}\left\{\phi_n(P)\left[M_{kn}^{*j}(q,P)-M_{kn}^{*a}(q,P)\right]+\phi_s(P)\left[M_{kns}^{*j}(q,P)-M_{kns}^{*a}(q,P)\right]+\right.$$
$$\left.+w(P)\left[Q_{kn}^{*j}(q,P)-Q_{kn}^{*a}(q,P)\right]\right\}d\Gamma+$$
$$+\sum_{j=1}^{N_s}\int_{\Gamma_1}\left[M_n(P)\phi_{kn}^{*j}(q,P)+M_{ns}(P)\phi_{ks}^{*j}(q,P)+Q_n(P)w_k^{*j}(q,P)\right]d\Gamma$$
$$+\sum_{j=1}^{N_{\text{int}}}\int_{\Gamma_{ja}}\left\{M_n(P)\left[\phi_{kn}^{*j}(q,P)-\phi_{kn}^{*a}(q,P)\right]+M_{ns}(P)\left[\phi_{ks}^{*j}(q,P)-\phi_{ks}^{*a}(q,P)\right]+\right.$$
$$\left.+Q_n(P)\left[w_k^{*1}(q,P)-w_k^{*a}(q,P)\right]\right\}d\Gamma+$$
$$+\sum_{j=1}^{N_{\text{int}}}\int_{\Gamma_{ja}}\left\{p_n(P)\left[c_j\phi_{kn}^{*j}(q,P)-c_a\phi_{kn}^{*a}(q,P)\right]+\right.$$
$$\left.+\sum_{j=1}^{N_{\text{int}}}\int_{\Gamma_{ja}}p_s(P)\left[c_j\phi_{ks}^{*j}(q,P)-c_a\phi_{ks}^{*a}(q,P)\right]\right\}d\Gamma+$$
$$+\sum_{j=1}^{N_s}\int_{\Gamma_1}c_j\left[p_n(P)\phi_{kn}^{*j}(q,P)+p_s(P)\phi_{ks}^{*j}(q,P)\right]d\Gamma \quad (15)$$

where *k=m, l, 3; i=1, 2* and all values (w, $_s$, $_n$, p_n, p_s, M_n, M_{ns} and Q_n) are referred to the reference surface.

Let us now consider the stretching problem. For simplicity and also to eliminate the in-plane tractions along the interfaces, the fundamental value $u_{ki}^{*(j)}$ related to the sub-region Ω_j will be written in terms of the value u_{ki}^* referred to the sub-region where the collocation point is placed as follow:

$$u_{ki}^{*(j)}=u_{ki}^*\overline{E}/\overline{E}_j \quad (16)$$

where $\overline{E}_j = E_j t_j$.

From the weighted residual method and considering (16) one can derive the integral representation of displacements for the sub-region Ω_j . The following integral representation for the whole plate is obtained by summing the integral equations of all sub-regions and enforcing equilibrium and compatibility conditions along the interface:

$$u_k(q) = -\sum_{i=1}^{N_S}\frac{\overline{E}_i}{\overline{E}}\int_{\Gamma_i}\left[u_n p^*_{kn} + u_s p^*_{ks}\right]d\Gamma - \sum_{m=1}^{N_{\text{int}}}\frac{\left(\overline{E}_j - \overline{E}_a\right)}{\overline{E}}\int_{\Gamma_{ja}}\left[u_n p^*_{kn} + u_s p^*_{ks}\right]d\Gamma_{ja} + \int_{\Gamma}\left[u^*_{kn} p_n + u^*_{ks} p_s\right]d\Gamma + \int_{\Omega}\left[u^*_{kn} b_n + u^*_{ks} b_s\right]d\Omega \qquad (17)$$

Note that in (17) the in-plane tractions were eliminated from the interfaces, where the only remaining values are the displacements.

Writing now the in-plane displacements defined over the middle surface (u_s and u_n) in terms of their values on the reference surface ($\boldsymbol{u_i^j} = \boldsymbol{u_i^r} - \boldsymbol{c_j}\phi_i$, with i=n,s) the following stretching integral equation for the coupled stretching-bending problem is obtained:

$$\left[-c(q)\phi_k(q) + u_k(q)\right] = -\sum_{i=1}^{N_S}\frac{\overline{E}_i}{\overline{E}}\int_{\Gamma}\left[u_n p^*_{kn} + u_s p^*_{ks}\right]d\Gamma + -\sum_{m=1}^{N_{\text{int}}}\frac{\left(\overline{E}_j - \overline{E}_a\right)}{\overline{E}}\int_{\Gamma_{ja}}\left[u_n p^*_{kn} + u_s p^*_{ks}\right]d\Gamma_{ja} + \sum_{i=1}^{N_S}\frac{\overline{E}_i}{\overline{E}}\int_{\Gamma} c_i\left[p^*_{kn}\phi_n + p^*_{ks}\phi_s\right]d\Gamma + \sum_{m=1}^{N_{\text{int}}}\frac{\left(\overline{E}_j c_j - \overline{E}_a c_a\right)}{\overline{E}}\int_{\Gamma_{ja}}\left(p^*_{kn}\phi_n + p^*_{ks}\phi_s\right)d\Gamma_{ja} + \int_{\Gamma}\left(u^*_{kn} p_n + u^*_{ks} p_s\right)d\Gamma + \int_{\Omega_i}\left(u^*_{kn} b_n + u^*_{ks} b_s\right)d\Omega \qquad (18)$$

In (18) are defined only displacements along the interfaces, given by: u_s, u_n, ${}_s$ and ${}_n$. Along the external boundary, besides the previous values are also defined the in-plane tractions p_n and p_s.

Let us now consider the beam B_3 represented in Fig. 2a by the sub-region Ω_3. In order to reduce the number of degrees of freedom, some Kinematic hypothesis will be assumed along the beams cross sections, leading to a model where the values will be defined along the beam skeleton line instead of its boundary. The displacements w, u_s, u_n, ${}_s$ and ${}_n$. will be assumed to be linear along the beam width. Thus the interface displacement vector related to the beam interfaces are translated to the skeleton line, as follows:

$$\phi_k^{\Gamma_{32}} = \phi_k + \phi_{k,n}\, b_3/2 \qquad \text{k=n,s} \qquad (19a)$$

$$\phi_k^{\Gamma_{31}} = -\left[\phi_k - \phi_{k,n}\, b_3/2\right] \qquad (19b)$$

$$w^{\Gamma_{32}} = w + w_{,n}\, b_3/2 \qquad (19c)$$

$$w^{\Gamma_{31}} = w - w_{,n}\, b_3/2 \qquad (19d)$$

$$u_k^{\Gamma_{32}} = u_k + u_{k,n}\, b_3/2 \qquad (20a)$$

$$u_k^{\Gamma_{31}} = -\left[u_k - u_{k,n}\, b_3/2\right] \qquad (20b)$$

where b_3 is the beam width , $\phi_k^{\Gamma_{ij}}$, $u_k^{\Gamma_{ij}}$ and $w^{\Gamma_{ij}}$ are displacement components along the interface Γ_{ij} ; ϕ_k , w $\phi_{k,n}$, $u_{k,n}$ and $w_{,n}$ are components along the skeleton line.

Observe that adopting the approximations defined in (19) and (20), new variables related to the beam axis appear in the formulation: the rotations $w_{,n}$ $u_{s,n}$ and $u_{n,n}$ and the curvatures ${}_{s,n}$ and ${}_{n,n}$ which will be all considered constant along the beam width.

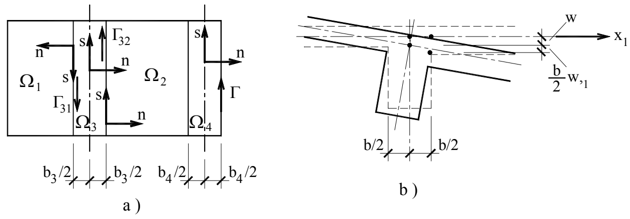

Fig. 2 (a) reinforced plate view; (b) deflection approximations along interfaces

The tractions M_n, M_{ns} and Q_n defined on the interfaces will be written in terms of its components along the beam axis as follow:

$$Q_n^{\Gamma_{32}} = -Q_n^{\Gamma_{31}} = Q_n \qquad (21)$$

$$M_n^{\Gamma_{32}} = M_n + Q_n b_3/2 \qquad (22)$$

$$M_n^{\Gamma_{31}} = M_n - Q_n b_3/2 \qquad (23)$$

$$M_{ns}^{\Gamma_{32}} = M_{ns}^{\Gamma_{31}} = M_{ns} \qquad (24)$$

where M_n, M_{ns} and Q_n refers to the beam axis while the directions of $M_n^{\Gamma_{ij}}$, $M_{ns}^{\Gamma_{ij}}$ and $Q_n^{\Gamma_{ij}}$ are given by the local coordinate system defined on interfaces.

Finally, the stress related to the stretching problem will be assumed to vary linearly along the beam cross section, resulting into the following approximations for the in-plane tractions (p_n and p_s):

$$p_i^{\Gamma_{32}} = \frac{1}{2}p_i = -p_i^{\Gamma_{31}} \qquad \text{i=n, s} \qquad (25)$$

where $\boldsymbol{p_i}$ is the resultant on the skeleton line.

Note that the integral representations of $w_{,m}$, ${}_{k,m}$ and $u_{k,m}$ that can be easily obtained by differentiating (15) or (18). Despite of the values being defined along the beam axis, the integrals are still performed on the interfaces. Thus as the collocation points are adopted on the beam axis there is no problem of singularities.

IV. Algebraic Equations

The integral representations (15) and (18) have to be transformed into algebraic expressions after discretizing the boundary and interfaces into elements. It has been adopted linear elements to approximate the problem geometry while the variables were approximated by quadratic shape functions.

Along the external boundary without beams ten values are defined (w, ϕ_n, ϕ_s, Q_n, M_n, M_{ns}, u_s; u_n; p_n and p_s), being five of them prescribed, requiring therefore five algebraic equations for each boundary node. It has been adopted to write (15) plus (18) for an external collocation point very near to the boundary.

For each external or internal beam node are defined fifteen values: $_{n}$, $_{s}$, $_{s,n}$; $_{n,n}$, w, $w_{,n}$, M_n, M_{ns}, Q_n u_s; u_n; $u_{s,n}$; $u_{n,n}$; p_n and p_s. All these values remain as unknowns in the internal beams, requiring therefore fifteen algebraic equations which will be written for collocation points on the skeleton line. In this case the adopted equations were those corresponding to the unknowns. For external beams, the displacements $u_{s,n}$; $u_{n,n}$; $_{s,n}$; $_{n,n}$ and $w_{,n}$ are problem unknowns while five of the remaining values must be prescribed, leading to ten unknowns for each external beam node. It has been adopted to write, for collocations points on the beam axis, the following ten algebraic equations: u_s; u_n; $u_{s,n}$; $u_{n,n}$; w, ϕ_n, ϕ_s, $_{s,n}$; $_{n,n}$ and $w_{,n}$. In both cases the collocations can be coincident with the chosen node or defined at element internal points when variable discontinuity is required at the element end.

After writing the recommended algebraic relations one obtains the set of equations defined bellow which can be solved after applying the boundary conditions.

$$\begin{bmatrix} [H]^B & [\overline{H}]^S \\ --- & --- \\ [\overline{H}]^B & [H]^S \end{bmatrix} \begin{Bmatrix} \{U\}_B \\ \{U\}_S \end{Bmatrix} = \begin{bmatrix} [G]^B & [\overline{G}]^S \\ --- & --- \\ [0] & [G]^S \end{bmatrix} \begin{Bmatrix} \{P\}_B \\ \{P\}_S \end{Bmatrix} + \begin{Bmatrix} \{T\}^B \\ --- \\ \{T\}^S \end{Bmatrix} \tag{26}$$

where the upper e bottom parts indicate, respectively, boundary and interface collocation algebraic equations of the bending and stretching problems; $\{U\}$and $\{P\}$ are displacement and traction vectors, respectively; $\{T\}$ is the independent vector due to the applied loads; $[H]$ and $[G]$ are matrices obtained by integrating all boundary and interfaces; B and S are related to bending and stretching problems.

Equation (26) can be represented in a reduced form, as follows:

$$\underset{\sim}{H}\,\underset{\sim}{U} = \underset{\sim}{G}\,\underset{\sim}{P} + \underset{\sim}{T} \tag{27}$$

V. Numerical Application

An example is now shown to demonstrate the performance of the proposed formulation: a simple plate reinforced by an internal beam with particular load to have exact solution.

Let us consider the stiffened plate depicted in Fig. 4a. In this example the two small sides are assumed simply supported, while the other two are free ($Q_n=M_n=M_{ns}=p_s=p_n=0$). The small side with coordinate x_1=0.0cm is fix (**u_n=0**), with the following boundary conditions: $w=M_n=M_{ns}=u_s=u_n=0$ for the middle point and $w=M_n=M_{ns}=p_s=u_n=\mathbf{0}$ for the remaining nodes. On the opposite side is prescribed the load $p_n=10000kN/m$**,** being he others prescribed values given by**:** $w=M_n=M_{ns}=p_s=0$**.** The plate and beam thicknesses are, respectively: $h_p=10cm$ and $h_B=20cm$. The following elastic parameters have been adopted: Young's modulus $E = 3.0x10^6 kN/m^2$ and Poisson's ratio $\nu=0.0$.

One has used 12 elements to discretize the plate sides without beams, 2 elements along the beam axe and 1 element on each boundary corresponding to the beam width, resulting into sixteen quadratic elements and forty one nodes (see Fig 4b). This poor discretization is enough to lead to the exact values for both displacements and tractions.

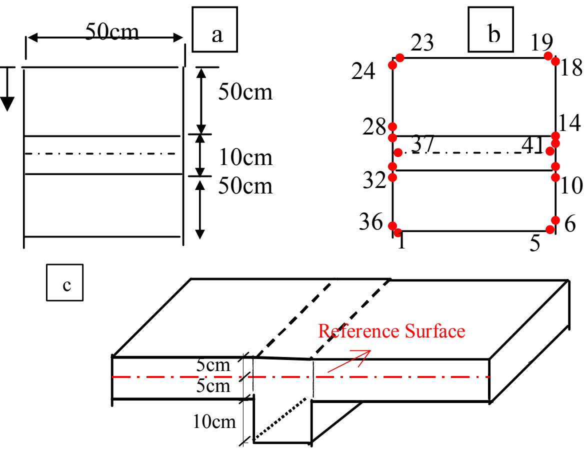

Fig. 4 Stiffened plate (a) geometry; (b) discretization, (c) plate view

The plate middle surface will be adopted as the reference one (see Fig. 4c), therefore $c_P=0.0cm$ and $c_B=5cm$ for the plate and beam. In any of the three sub-regions the exact solution for displacement in the x_1 direction is given by: $\Delta u_1 = P_n x_1\left(-c_j^2 / I + 1 / A\right)/ E$, being c_j the value of c in the sub-region Ω_j . In Table I are shown the analytical and numerical results for the displacement u_1, considering two discretizations: the one defined in Fig. 4 and a finer mesh with 58 elements.

TABLE I
NUMERICAL AND EXACT RESULTS IN BOUNDARY NODES

	u_1 (cm)Node 14 (x_1=50cm)	u_1 (cm)Node 10 (x_1=60cm)	u_1 (cm) Node 6 (x_1=110c)
Exact solution	1,6667	1,83334	3,5
16 elements	1,668	1,836	3,50
58 elements	1,667	1,834	3,50

Moreover the computed in-plane traction P_n was equal to the applied load for both the nodes on the fixed side and the internal beam axis, as expected.

VI. CONCLUSION

The BEM formulation based on Reissner's hypothesis for bending analysis of plates reinforced by beams has been extended to take into account the membrane effects. The beams are treated as narrow sub-regions with larger thickness and are not displayed over their middle surface. The performance of the proposed formulation has been confirmed by comparing the results with analytical solutions.

REFERENCES

[1] Hu, C. & Hartley, G.A., Elastic analysis of thin plates with beam supports. *Engineering Analysis with Boundary Elements*, **13**: 229-238, 1994.
[2] Tanaka, M. & Bercin, A.N., A boundary Element Method applied to the elastic bending problems of stiffened plates. In: *Boundary Element Method XIX*, Eds. C.A. Brebbia et al., CMP, Southampton, 1997.
[3] Sapountzakis, E.J. & Katsikadelis, J.T., Analysis of plates reinforced with beams. *Computational Mechanics*, 26: 66-74, 2000.
[4] Fernandes, G.R and Venturini, W.S., Stiffened plate bending analysis by the boundary element method. *Computational Mechanics*, 28: 275-281, 2002.
[5] Fernandes, G.R. and Venturini, W. S., *Building floor analysis by the Boundary element method*. Computational Mechanics, 35:277-291, 2005.
[6] Hartmann, F. and Zotemantel, R., The direct boundary element method in plate bending. *International Journal for Numerical Methods in Engineering*, 23(11): 2049-2069, 1986.
[7] Kirchhoff, G. (1850). Uber das gleichgewicht und die bewegung einer elastischen scleibe. *J. Math.*, n.40, p.51-58.
[8] Reissner, E. (1947). *On bending of elastic plates.* Quart. Appl. Math.; 5(1); 55-68; 1947.
[9] Weën, F. V. (1982). *Application of boundary integral equation method to Reissner's plate model*. Int. J. Num. Meth. Emg., v. 18 (1), p. 1-10.
[10] Palermo Jr. L., Plate bending analysis using the classical or the Reissner-Mindlin models. *Engineering Analysis with Boundary Elements*, 27: 603-609, 2003.