

# The Number of Rational Points on Singular Curves $y^2 = x(x - a)^2$ over Finite Fields $\mathbf{F}_p$

Ahmet Tekcan

**Abstract**—Let  $p \geq 5$  be a prime number and let  $\mathbf{F}_p$  be a finite field. In this work, we determine the number of rational points on singular curves  $E_a : y^2 = x(x - a)^2$  over  $\mathbf{F}_p$  for some specific values of  $a$ .

**Keywords**—Singular curve, elliptic curve, rational points.

## I. INTRODUCTION

Mordell began his famous paper [9] with the words Mathematicians have been familiar with very few questions for so long a period with so little accomplished in the way of general results, as that of finding the rational points on elliptic curves. The history of elliptic curves is a long one, and exciting applications for elliptic curves continue to be discovered. Recently, important and useful applications of elliptic curves have been found for cryptography [4], [7], [8], for factoring large integers [6] and for primality proving [1], [3]. The mathematical theory of elliptic curves was also crucial in the proof of Fermat's Last Theorem [17].

Let  $q$  be a positive integer,  $\mathbf{F}_q$  be a finite field and let  $\overline{\mathbf{F}}_q$  denote the algebraic closure of  $\mathbf{F}_q$  with  $\text{char}(\overline{\mathbf{F}}_q) \neq 2, 3$ . An elliptic curve  $E$  over  $\mathbf{F}_q$  is defined by an equation

$$E : y^2 = x^3 + ax^2 + bx,$$

where  $a, b \in \mathbf{F}_q$  and  $b^2(a^2 - 4b) \neq 0$ . The discriminant of  $E$  is

$$\Delta = 16b^2(a^2 - 4b).$$

If  $\Delta = 0$ , then  $E$  is not an elliptic curve is a singular curve. We can view an elliptic curve  $E$  as a curve in projective plane  $\mathbf{P}^2$ , with a homogeneous equation

$$y^2z = x^3 + ax^2z + bxz^2,$$

and one point at infinity, namely  $(0, 1, 0)$ . This point  $\infty$  is the point where all vertical lines meet. We denote this point by  $O$ . Let

$$E(\mathbf{F}_q) = \{(x, y) \in \mathbf{F}_q \times \mathbf{F}_q : y^2 = x^3 + ax^2 + bx\} \cup \{O\}$$

denote the set of rational points  $(x, y)$  on  $E$ . Then it is a subgroup of  $E$ . The order of  $E(\mathbf{F}_q)$ , denoted by  $N_q = \#E(\mathbf{F}_q)$ , is defined as the number of the rational points on  $E$  and is given by

$$\#E(\mathbf{F}_q) = q + 1 + \sum_{x \in \mathbf{F}_q} \left( \frac{x^3 + ax^2 + bx}{\mathbf{F}_q} \right),$$

where  $\left( \frac{\cdot}{\mathbf{F}_q} \right)$  denotes the Legendre symbol (for further details on elliptic curves see [10], [11], [16]).

Ahmet Tekcan is with the Uludag University, Department of Mathematics, Faculty of Science, Bursa-TURKEY, email: tekcan@uludag.edu.tr, <http://matematik.uludag.edu.tr/AhmetTekcan.htm>.

## II. THE NUMBER OF RATIONAL POINTS ON SINGULAR CURVES $y^2 = x(x - a)^2$ OVER $\mathbf{F}_p$ .

In [2], [12], [14], we considered some specific elliptic curves (including the number of rational points on them) over finite fields. In this section we will determine the number of rational points on singular curves

$$E_a : y^2 = x(x - a)^2 \quad (1)$$

over finite fields  $\mathbf{F}_p$  for primes  $p \geq 5$ . Let

$$E_a(\mathbf{F}_p) = \{(x, y) \in \mathbf{F}_p \times \mathbf{F}_p : y^2 = x(x - a)^2\} \cup O.$$

Before we consider our problem we give some notations which we need them later.

**Lemma 2.1:** [5] Let  $p$  be an odd prime and let  $f(x) \in \mathbf{Z}[x]$  be a polynomial of degree  $\geq 1$ . Then the number  $N_p(f)$  of solutions  $(x, y) \in \mathbf{F}_p \times \mathbf{F}_p$  of the congruence  $y^2 \equiv f(x) \pmod{p}$  is

$$N_p(f) = p + S_p(f), \quad (2)$$

where

$$S_p(f) = \sum_{x=0}^{p-1} \left( \frac{f(x)}{p} \right). \quad (3)$$

Also it is showed in [16] that for the polynomial  $f(x) = (x - r)^2(x - s)$  of degree 3 for some  $r, s \in \mathbf{F}_p$ ,

$$\sum_{x=0}^{p-1} \left( \frac{f(x)}{\mathbf{F}_p} \right) = - \left( \frac{r - s}{\mathbf{F}_p} \right). \quad (4)$$

Note that the  $f(x) = x(x - a)^2$  is a polynomial of degree 3. So by considering the point 0, we can rewrite the formula (2) as

$$\begin{aligned} \#E_a(\mathbf{F}_p) &= p + 1 + \sum_{x=0}^{p-1} \left( \frac{x(x - a)^2}{p} \right) \\ &= p + 1 - \left( \frac{a}{p} \right) \end{aligned} \quad (5)$$

by (3) and (4). Therefore if  $\left( \frac{a}{p} \right) = 1$ , then  $\#E_a(\mathbf{F}_p) = p$  and if  $\left( \frac{a}{p} \right) = -1$ , then  $\#E_a(\mathbf{F}_p) = p + 2$ . Therefore the order of  $E_a$  over  $\mathbf{F}_p$  is depends on whether  $a$  is a quadratic residue or not.

Now we can give the following two theorems which I proved them in [13] and [15], respectively.

**Theorem 2.1:** Let  $\mathbf{F}_p$  be a finite field. Then

$$\begin{aligned} \left(\frac{1}{p}\right) &= 1 \text{ for every primes } p \geq 5 \\ \left(\frac{2}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 7(8) \\ -1 & \text{if } p \equiv 3, 5(8) \end{cases} \\ \left(\frac{3}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 11(12) \\ -1 & \text{if } p \equiv 5, 7(12) \end{cases} \\ \left(\frac{4}{p}\right) &= 1 \text{ for every primes } p \geq 5 \\ \left(\frac{5}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 9(10) \\ -1 & \text{if } p \equiv 3, 7(10) \end{cases} \\ \left(\frac{6}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 5, 19, 23(24) \\ -1 & \text{if } p \equiv 7, 11, 13, 17(24) \end{cases} \\ \left(\frac{7}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 3, 9, 19, 25, 27(28) \\ -1 & \text{if } p \equiv 5, 11, 13, 15, 17, 23(28) \end{cases} \\ \left(\frac{8}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 7, 17, 23(24) \\ -1 & \text{if } p \equiv 5, 11, 13, 19(24) \end{cases} \\ \left(\frac{9}{p}\right) &= 1 \text{ for every primes } p \geq 11 \\ \left(\frac{10}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 3, 9, 13, 27, 31, 37, 39(40) \\ -1 & \text{if } p \equiv 7, 11, 17, 19, 21, 23, 29, 33, 37(40). \end{cases} \end{aligned}$$

**Theorem 2.2:** Let  $\mathbf{F}_p$  be a finite field. Then

$$\begin{aligned} \left(\frac{-1}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1(4) \\ -1 & \text{if } p \equiv 3(4) \end{cases} \\ \left(\frac{-2}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 3(8) \\ -1 & \text{if } p \equiv 5, 7(8) \end{cases} \\ \left(\frac{-3}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 7(12) \\ -1 & \text{if } p \equiv 5, 11(12) \end{cases} \\ \left(\frac{-4}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 5(12) \\ -1 & \text{if } p \equiv 7, 11(12) \end{cases} \\ \left(\frac{-5}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 3, 7, 9(20) \\ -1 & \text{if } p \equiv 11, 13, 17, 19(20) \end{cases} \\ \left(\frac{-6}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 5, 7, 11, 25, 29, 31, 35(48) \\ -1 & \text{if } p \equiv 13, 17, 19, 23, 37, 41, 43, 47(48) \end{cases} \\ \left(\frac{-7}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 9, 11, 15, 23, 25(28) \\ -1 & \text{if } p \equiv 3, 5, 13, 17, 19, 27(28) \end{cases} \\ \left(\frac{-8}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 11, 17, 19, 25, 35, 41, 43(48) \\ -1 & \text{if } p \equiv 5, 7, 13, 23, 29, 31, 37, 47(48) \end{cases} \\ \left(\frac{-9}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 5, 13, 17(24) \\ -1 & \text{if } p \equiv 7, 11, 19, 23(24) \end{cases} \\ \left(\frac{-10}{p}\right) &= \begin{cases} 1 & \text{if } p \equiv 1, 7, 9, 11, 13, 19, 23, 37(40) \\ -1 & \text{if } p \equiv 3, 17, 21, 27, 29, 31, 33, 39(40). \end{cases} \end{aligned}$$

Now we can consider our main problem.

**Theorem 2.3:** Let  $E_a$  be the singular curve defined in (1). Then

$$\begin{aligned} \#E_1(\mathbf{F}_p) &= p \text{ for every primes } p \geq 5 \\ \#E_2(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 7(8) \\ p+2 & \text{if } p \equiv 3, 5(8) \end{cases} \\ \#E_3(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 11(12) \\ p+2 & \text{if } p \equiv 5, 7(12) \end{cases} \\ \#E_4(\mathbf{F}_p) &= p \text{ for every primes } p \geq 5 \\ \#E_5(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 9(10) \\ p+2 & \text{if } p \equiv 3, 7(10) \end{cases} \\ \#E_6(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 5, 19, 23(24) \\ p+2 & \text{if } p \equiv 7, 11, 13, 17(24) \end{cases} \\ \#E_7(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 3, 9, 19, 25, 27(28) \\ p+2 & \text{if } p \equiv 5, 11, 13, 15, 17, 23(28) \end{cases} \\ \#E_8(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 7, 17, 23(24) \\ p+2 & \text{if } p \equiv 5, 11, 13, 19(24) \end{cases} \\ \#E_9(\mathbf{F}_p) &= p \text{ for every primes } p \geq 11 \\ \#E_{10}(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 3, 9, 13, 27, 31, 37, 39(40) \\ p+2 & \text{if } p \equiv 7, 11, 17, 19, 21, 23, 29, 33, 37(40) \end{cases} \\ \#E_{-1}(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1(4) \\ p+2 & \text{if } p \equiv 3(4) \end{cases} \\ \#E_{-2}(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 3(8) \\ p+2 & \text{if } p \equiv 5, 7(8) \end{cases} \\ \#E_{-3}(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 7(12) \\ p+2 & \text{if } p \equiv 5, 11(12) \end{cases} \\ \#E_{-4}(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 5(12) \\ p+2 & \text{if } p \equiv 7, 11(12) \end{cases} \\ \#E_{-5}(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 3, 7, 9(20) \\ p+2 & \text{if } p \equiv 11, 13, 17, 19(20) \end{cases} \\ \#E_{-6}(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 5, 7, 11, 25, 29, 31, 35(48) \\ p+2 & \text{if } p \equiv 13, 17, 19, 23, 37, 41, 43, 47(48) \end{cases} \\ \#E_{-7}(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 9, 11, 15, 23, 25(28) \\ p+2 & \text{if } p \equiv 3, 5, 13, 17, 19, 27(28) \end{cases} \\ \#E_{-8}(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 11, 17, 19, 25, 35, 41, 43(48) \\ p+2 & \text{if } p \equiv 5, 7, 13, 23, 29, 31, 37, 47(48) \end{cases} \\ \#E_{-9}(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 5, 13, 17(24) \\ p+2 & \text{if } p \equiv 7, 11, 19, 23(24) \end{cases} \\ \#E_{-10}(\mathbf{F}_p) &= \begin{cases} p & \text{if } p \equiv 1, 7, 9, 11, 13, 19, 23, 37(40) \\ p+2 & \text{if } p \equiv 3, 17, 21, 27, 29, 31, 33, 39(40). \end{cases} \end{aligned}$$

*Proof:* Applying Theorems 2.1 and 2.2 the result is clear. ■

Now we consider the sum of  $x$ - and  $y$ -coordinates of all rational points  $(x, y)$  on  $E_a$  over  $F_p$ . Let  $[x]$  and  $[y]$  denote the  $x$ - and  $y$ -coordinates of the points  $(x, y)$  on  $E_a$ , respectively. Then we have the following the results.

*Theorem 2.4:* The sum of  $[x]$  on  $E_a$  is

$$\sum_{[x]} E_a(\mathbf{F}_p) = \begin{cases} \frac{p^3-p-12a}{12} & \text{if } (\frac{a}{p}) = 1 \\ \frac{p^3-p+12a}{12} & \text{if } (\frac{a}{p}) = -1. \end{cases}$$

*Proof:* Let  $U_p = \{1, 2, \dots, p-1\}$  be the set of units in  $\mathbf{F}_p$ . Then then taking squares of elements in  $U_p$ , we would obtain the set of quadratic residues  $Q_p = \{1^2, 2^2, \dots, (\frac{p-1}{2})^2\}$ . Then the sum of all elements in  $Q_p$  hence

$$\sum_{x \in Q_p} x = \frac{p^3-p}{24}.$$

Now let  $(\frac{a}{p}) = 1$ . Then  $a$  is a quadratic residue. But for this values of  $a$ , there is one rational point  $(a, 0)$  on  $E_a$ . Let  $H = Q_p - \{a\}$ . Then

$$\begin{aligned} \sum_{x \in H} x &= \left( \sum_{x \in Q_p} x \right) - a \\ &= \frac{p^3-p}{24} - a \\ &= \frac{p^3-p-24a}{24}. \end{aligned}$$

We know that every element  $x$  of  $H$  makes  $x(x-a)^2$  is a square. Let  $x(x-a)^2 \equiv t^2 \pmod{p}$ . Then  $y^2 \equiv t^2 \pmod{p}$ . So there are two rational points  $(x, t)$  and  $(x, p-t)$  on  $E_a$ . The sum of  $x$ -coordinates of these two points is  $2x$ , that is, for every  $x \in H$ , the sum of  $x$ -coordinates of  $(x, t)$  and  $(x, p-t)$  is  $2x$ . So the sum of  $x$ -coordinates of all points on  $E_a$  is

$$2 \sum_{x \in H} x.$$

Further we said above that the point  $(a, 0)$  is also on  $E_a$ . Consequently

$$\sum_{[x]} E_a(\mathbf{F}_p) = 2 \left( \sum_{x \in H} x \right) + a = \frac{p^3-p-12a}{12}.$$

Let  $(\frac{a}{p}) = -1$ . Then  $a$  is not a quadratic residue. But every element  $x$  of  $Q_p$  makes  $x(x-a)^2$  a square. So there are two rational points on  $E_a$  and hence the sum of  $x$ -coordinates of these two points is  $2x$ . Further  $(a, 0)$  is also a rational point on  $E_a$ . Consequently

$$\sum_{[x]} E_a(\mathbf{F}_p) = 2 \left( \sum_{x \in Q_p} x \right) + a = \frac{p^3-p+12a}{12}.$$

*Theorem 2.5:* The sum of  $[y]$  on  $E_a$  is

$$\sum_{[y]} E_a(\mathbf{F}_p) = \begin{cases} \frac{p^2-3p}{2} & \text{if } (\frac{a}{p}) = 1 \\ \frac{p^2-p}{2} & \text{if } (\frac{a}{p}) = -1. \end{cases}$$

*Proof:* Let  $(\frac{a}{p}) = 1$ . Then  $a$  is a quadratic residue but again for this value of  $a$ , there is one rational point  $(a, 0)$  on  $E_a$ . Also every element  $x$  of  $Q_p$  makes  $x(x-a)^2$  a square. Let  $x(x-a)^2 \equiv t^2 \pmod{p}$ . Then

$$y^2 \equiv t^2 \pmod{p} \Leftrightarrow y \equiv \pm t \pmod{p}.$$

So there are two points  $(x, t)$  and  $(x, p-t)$  on  $E_a$ . The sum of  $y$ -coordinates of these two points is  $p$ . We know that there are  $\frac{p-1}{2} - 1 = \frac{p-3}{2}$  points  $x$  such that  $x(x-a)^2$  is a square. So the sum of  $y$ -coordinates of all points  $(x, y)$  on  $E_a$  is

$$p \left( \frac{p-3}{2} \right) = \frac{p^2-3p}{2}.$$

Now let  $(\frac{a}{p}) = -1$ . Then  $a$  is not a quadratic residue. But every element  $x$  of  $Q_p$  makes  $x(x-a)^2$  a square. Let  $x(x-a)^2 \equiv t^2 \pmod{p}$ . Then

$$y^2 \equiv t^2 \pmod{p} \Leftrightarrow y \equiv \pm t \pmod{p}.$$

So there are two points  $(x, t)$  and  $(x, p-t)$  on  $E_a$ . The sum of  $y$ -coordinates of these two points is  $p$ . We know that there are  $\frac{p-1}{2}$  points  $x$  in  $Q_p$  such that  $x(x-a)^2$  is a square. So the sum of  $y$ -coordinates of all points  $(x, y)$  on  $E_a$  is

$$p \left( \frac{p-1}{2} \right) = \frac{p^2-p}{2}.$$

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